Continuous Dynamical Systems

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[1]based exclusively on notes given by G. R. Hall in MA 775, Fall 1996, at Boston University

Contents

Introduction

In Fall 1996, Professor G. R. Hall gave a class at Boston University titled "MA 775 - Ordinary Differential Equations". The following pages are a (hopefully accurate) copy of the notes I took in this class. He deserves all the credit for assembling the material and presenting it. The only mathematics I ended up doing was checking and making sure I had all of the inequalities pointing the right way, etc.

This document was produced using $\mathcal{AMS}$-LaTeX macros on top of Version 3.1415 of the LaTeX2ϵ compiler. I used the amsbook documentclass. The only packages I had to import were epsfig and graphicx for the pictures, and theorem to define the various theorem environments. All of the pictures were made either with `xfig` or Mathematica, and then dumped into PostScript[1].

For further information (including how to get a copy of this PostScript file), please go to `http://math.bu.edu/people/deville/Notes/`. The latest versions of this file will be maintained there when corrections are made. If you find any errors, or have any general comments on the notes, please send email to me at `deville@math.bu.edu`. I will maintain an errata page for these notes (and all future releases), and any and all comments are greatly appreciated.

[1] In this particular version, a few of the pictures are still not in. They tend to be more at the end, and most of them are concerning the Melnikov estimates. I plan to do the rest eventually, but they're hard and complicated, so relax.

CHAPTER 1

Introduction

1. Some preliminary definitions (Sept. 4)

DEFINITION 1.1. *Given a vector field* $f\colon \mathbf{R}^n \to \mathbf{R}^m$ *then the* ***differential equation associated with*** f *is*

$$\dot{x} = f(x), \frac{dx}{dt} = f(x), x = (x_1, x_2, \ldots, x_n). \tag{1}$$

DEFINITION 1.2. *A* ***solution*** *is a curve* $\gamma\colon \mathbf{R} \to \mathbf{R}^n$ *such that*

$$\dot{\gamma}(t) = \frac{d\gamma}{dt} = f(\gamma(t)), \quad \textit{for all } t \tag{2}$$

DEFINITION 1.3. *An* ***initial condition*** *is* $x_0 \in \mathbf{R}^n$. *An* ***initial value problem***

$$\begin{cases} \dot{x} = f(x), \\ x(0) = x_0 \end{cases} \tag{3}$$

such that the solution is $\gamma(t)$ *with* $\gamma(0) = x_0$.

DEFINITION 1.4. *A* ***flow*** *is a map* $\phi\colon \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ *such that*

$$\phi(0, x_0) = x_0, \quad \forall x_0 \in \mathbf{R}^n \tag{4}$$

$$\phi(s+t, x_0) = \phi(s, \phi(t, x_0)) \tag{5}$$

Equation (5) is known as the ***group property****.*
A flow ϕ *is a solution of the differential equation* $\dot{x} = f(x)$ *if*

$$\frac{\partial \phi}{\partial t}(t, x_0) = f(\phi(t, x_0)), \quad \textit{for all } t, x_0. \tag{6}$$

Example: If $\dot{x} = x$ on $\mathbf{R}$, then we have

$$x(t) = x_o e^t, x(0) = x_0.$$
$$\phi(t, x_0) = x_0 e^t.$$

Claim: This is a flow.
Proof: (4) is simple to check.
For (5), we see that

$$\phi(s+t, x_0) = x_o e^{s+t},$$

and

$$\phi(s, \phi(t, x_0) = \phi(s, x_o e^t) = (x_0 e^t) e^s = x_o e^{s+t}.$$

□

The group property says that the "rules of evolution" (i.e. the vector field) do not change with time, or $f\colon \mathbf{R}^n \to \mathbf{R}^n$ does not depend on t. Such equations are called **autonomous**.

A "good theorem" could be that every "reasonable" differential equation has (almost) flow for solution, and flow is "nice".

THEOREM 1.1. *Given a smooth ($\mathcal{C}^\infty$) flow $\phi\colon \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}^n$, it is the solution of a differential equation, i.e. there exists a vector field $f\colon \mathbf{R}^n \to \mathbf{R}^n$ with ϕ as solution.*

Proof: Define $f(x_0) = \frac{\partial}{\partial t}\phi(0, x_0)$. Check that ϕ is a solution for $\dot{x} = f(x)$, i.e.

$$\begin{aligned} \frac{\partial}{\partial t}\phi(t, x_0) &= f(\phi(t, x_0)) \qquad \text{for all } t, x_0. \text{ Now, since} \\ \phi(t+\Delta t, x_0) &= \phi(\Delta t, \phi(t, x_0)), \end{aligned}$$

then

$$\begin{aligned} \frac{\partial}{\partial t}\phi(t, x_0) &= \lim_{\Delta t\to 0} \frac{\phi(\Delta t, \phi(t, x_0)) - \phi(0, \phi(t, x_0))}{\Delta t} \\ &= \frac{\partial}{\partial t}\phi(0, \phi(t, x_0)) \\ &\overset{\text{def}}{=} f(\phi(t, x_0)). \end{aligned}$$

□

Example:

$$\ddot{\theta} = -\sin(\theta)$$

Convert to

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = -\sin\theta \end{cases}$$

Example:

$$\ddot{y} + 3\dot{y} + 2y = \cos(2t)$$

The $3\dot{y}$ term is damping, the $2y$ term is Hooke's Law, and the $\cos(2t)$ term is an external forcing term.

Convert this to

$$\begin{cases} \dot{y} = v \\ \dot{v} = -3v - 2y + \cos(2t) \end{cases}$$

This is not autonomous. Using a new time variable, we get

$$\begin{cases} \dfrac{dy}{ds} = v \\[2ex] \dfrac{dy}{ds} = -3v - 2y + \cos(2t) \\[2ex] \dfrac{dt}{ds} = 1 \end{cases}$$

Be careful with the initial conditions in this case, since $t(s) = s + c$.

Example: A worse example. Consider $\dfrac{dx}{dt} = x^2$. If we solve, we get

$$x(t) = -\frac{1}{t + C}$$

We also get $C = -\dfrac{1}{x_0}$.

We get the flow

$$\phi(t, x_0) = -\frac{1}{t - \dfrac{1}{x_0}}$$

We can check that this is a flow, but the problem is the solution goes to ∞ in finite time, and is only defined for $t \in (-\infty, \dfrac{1}{x_0})$.

2. More examples, Changing variables (Sept. 6)

Remark: Think of $\phi(\cdot, x_0)\colon t \mapsto \phi(t, x_0)$ as starting at x_0, and going as far as it can. The advantage of thinking of this as a flow is that we can fix a time T and ask what happens to all (or a large set) of initial conditions after time T.

Continuity with respect to initial conditions says that for any fixed t, if x_0 and x are sufficiently close, then $\phi(t, x_0)$ and $\phi(t, x)$ are close.

Our earlier example $\dot{x} = x^2$ has a solution "flow"

$$\phi(t, x_0) = -\frac{x_0}{x_0 t - 1}.$$

Definition 2.1. *A* ***local flow*** *is a map $\phi\colon U \to \mathbf{R}^n$ where U is an open subset of $\mathbf{R} \times \mathbf{R}^n$ where, for each $x_0 \in \mathbf{R}^n$,*

$$(-\infty, \infty) \times \{x_0\} \cap U = (-a, b) \times \{x_0\}$$

and

$$\phi(0, x_0) = x_0 \tag{7}$$

$$\phi(s + t, x_0) = \phi(s, \phi(t, x_0)) \tag{8}$$

whenever both sides are defined.

Example: The solution of $\dot{x} = x^2$ has local flow

$$\phi(t, x_0) = -\frac{x_0}{x_0 t - 1},$$

defined on

$$(t, x_0), \text{ where } \begin{cases} -\infty < t < 1/x_0 & x_0 > 0 \\ -\infty < t < \infty & x_0 = 0 \\ 1/x_0 < t < \infty & x_0 < 0 \end{cases}$$

DEFINITION 2.2. *A **semiflow** is $\phi\colon \mathbf{R}^+ \to \mathbf{R}^n$ satisfying*

$$\phi(0, x_0) = x_0 \tag{9}$$

$$\phi(s+t, x_0) = \phi(s, \phi(t, x_0)) \tag{10}$$

where they are defined.

Example: The Kepler Problem. Let

$\overline{q}$ be the difference in position of 2 masses

$$\ddot{\overline{q}} = \frac{-G\mu\overline{q}}{\|\overline{q}\|^3}$$

where G is the gravity constant and μ is a constant related to the masses of the objects.

Make this a first order system ($\overline{q} = (q_1, q_2)$):

$$\begin{cases} \dot{q}_1 = p_1 \\ \dot{q}_2 = p_2 \\ \dot{p}_1 = \dfrac{-G\mu q_1}{(\sqrt{q_1^2 + q_2^2})^3} \\ \\ \dot{p}_2 = \dfrac{-G\mu q_2}{(\sqrt{q_1^2 + q_2^2})^3} \end{cases} \tag{11}$$

This is not a vector field on all of $\mathbf{R}^4$, because

$$\frac{1}{\left(\sqrt{q_1^2 + q_2^2}\right)^3} \to \infty \text{ as } q_1, q_2 \to 0$$

The **phase space** for this problem is $\mathbf{R}^4 \setminus \underbrace{\{(0,0,p_1,p_2)\}}_{\text{“the collision set”}}$. There are solutions which approach the collision set in finite time.

We can define a differential equation on any object where you have a good notion of tangent vector, such as $\mathbf{R}^n$, open subsets of $\mathbf{R}^n$, smooth surfaces on $\mathbf{R}^n$, or manifolds (which look locally like $\mathbf{R}^n$).

DEFINITION 2.3. *Given a flow ϕ or a differential equation $\dot{x} = f(x)$, the **orbit** of x_0 is*

$$\mathcal{O}(x_0) = \{\phi(t, x_0) : \phi \textit{ defined at } t\}$$

i.e., the image of the solution curve through x_0.

*We also define the **forward orbit** of x_0:*

$$\mathcal{O}^+(x_0) = \{\phi(t, x_0) : t \geq 0\}$$

DEFINITION 2.4. *A* ***fixed point, rest point, critical point,*** *or* ***stationary point,*** *is a point x_0 such that $\phi(t, x_0) = x_0 \quad \forall t$.*

x_0 is called a ***periodic point*** *if $\phi(x_0, T) = x_0$ for some $T > 0$ but $\phi(x_0, t) \neq x_0$ for $0 < t < T$. T is called the* ***least period****.*

Philosophy: For ODE we think of studying flows which come from nice vector fields. Tools we use are calculus tools (local analysis) to build global picture of flow.

When are two flows (or differential equations) "the same"?

Given differential equation, vector field $f : U \to \mathbf{R}^n$.

Suppose U has coordinates $X = (x_1, x_2, \ldots, x_n)$, and V has coordinates $Y = (y_1, y_2, \ldots, y_n)$.

The differential equation on U is $\dot{X} = f(X)$.

Given a smooth homeomorphism $h : V \to U$ which is smoothly invertible (i.e. h has a sufficient number of partials, and so does h^{-1}), we can define a vector field and a flow on V.

Move vector field, define $g : V \to \mathbf{R}^n$,

$$g(Y) = (Dh^{-1})\big|_{h(Y)} (f(h(Y))) \tag{12}$$

where Dh^{-1} is the derivative of $h^{-1} : U \to V$.

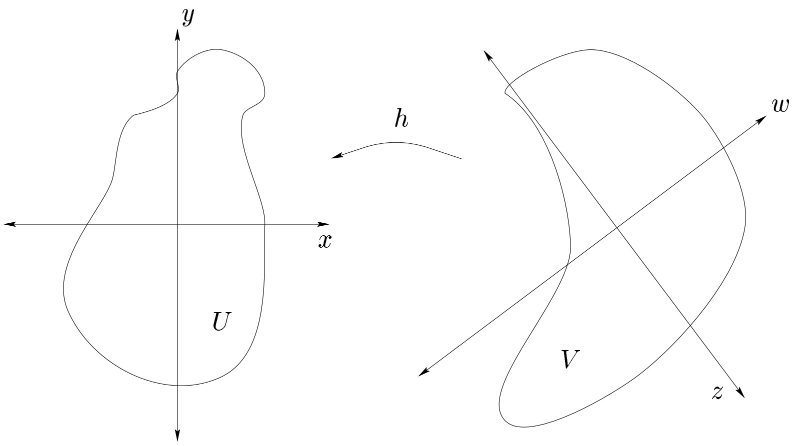

FIGURE 1. The map h.

3. Differentiation, change of time variable (Sept. 9)

The derivative is the "best linear approximation". To get a notion of the best linear approximation, we need a linear space. Start at $x \in U$, and $h(x) \in V$. We denote the tangent space to U at x as $T_x U$. The derivative of h is a linear map from $T_x U$ to $T_{h(x)} V$., i.e.

$$Dh|_x \, (v) = w, v \in T_x U, w \in T_{h(x)} V$$

To compute $Dh|_v$, pick $v \in T_x U$. Think of it as a velocity vector of a curve on U through x. Call this curve $\gamma(t)$, then $h(\gamma(t))$ is a curve on V. The velocity vector of $h(\gamma(t))$ at $h(x)$ is velocity of the image of γ on V.

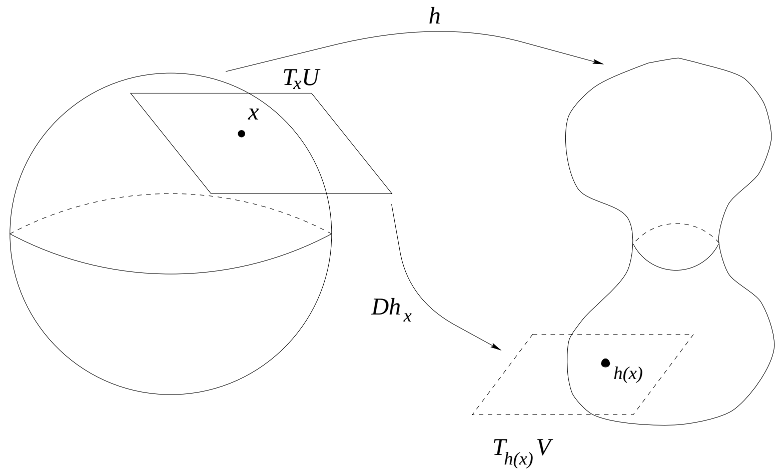

FIGURE 2. The derivative maps tangent planes to tangent planes.

So, define $Dh|_x\ (v) = w$ where w is the velocity vector of $h(\gamma(t))$ at $h(x)$. If

$$h(x_1, x_2, \ldots, x_n) = \\ (h_1(x_1, x_2, \ldots, x_n), h_2(x_1, x_2, \ldots, x_n), \ldots, h_n(x_1, x_2, \ldots, x_n)),$$

then

$$Dh|_x = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial x_1} & \cdots & \frac{\partial h_n}{\partial x_n} \end{bmatrix}$$

For example, start at x and move with velocity $(1, 0, 0, \ldots, 0)$:

$$h(x_1 + \Delta t, x_2, \ldots, x_n) - h(x_1, x_2, \ldots, x_n) \\ = \frac{(h_1(x_1 + \Delta t, x_2, \ldots, x_n) - h_1(x_1, x_2, \ldots, x_n), h_2 \ldots)}{\Delta t}$$

We get

$$\left(\frac{\partial h_1}{\partial x_1}, \frac{\partial h_2}{\partial x_1}, \ldots, \frac{\partial h_n}{\partial x_1}\right) = Dh|_x \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Example: $h\colon \mathbf{R}^2 \to \mathbf{R}^2$

$$\begin{aligned} h(x,y) &= (x^2+2xy, y+\cos(x)) = (z,w) \\ h(1,1) &= (3, 1+\cos(1)) \\ Dh &= \begin{pmatrix} 2x+2y & 2x \\ -\sin(x) & 1 \end{pmatrix} \\ Dh_{(1,1)} &= \begin{pmatrix} 4 & 2 \\ -\sin(1) & 1 \end{pmatrix} \\ Dh_{(1,1)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 4 \\ -\sin(1) \end{pmatrix} \end{aligned}$$

Let $\mathcal{L}(X \to Y)$ denote the linear maps from X to Y. If $h\colon \mathbf{R}^2 \to \mathbf{R}^2$, then

$$Dh\colon \mathbf{R}^2 \to \mathcal{L}(\mathbf{R}^2 \to \mathbf{R}^2) \cong \mathbf{R}^4$$
$$D^2h\colon \mathbf{R}^2 \to \mathcal{L}(\mathbf{R}^2 \to \mathcal{L}(\mathbf{R}^2 \to \mathbf{R}^2)) \cong \mathcal{L}(\mathbf{R}^2 \to \mathbf{R}^4) \cong \mathbf{R}^8$$

Back to changing variables...

Let $U \subseteq \mathbf{R}^n$ be open, and $f\colon U \to \mathbf{R}^n$ be a vector field. Then $\dot{x} = f(x)$ is a differential equation. Let x_0 be an initial condition. Then we have a solution curve $x(t)$, and also a solution (local) flow $\phi\colon \mathbf{R} \times U \to U$.

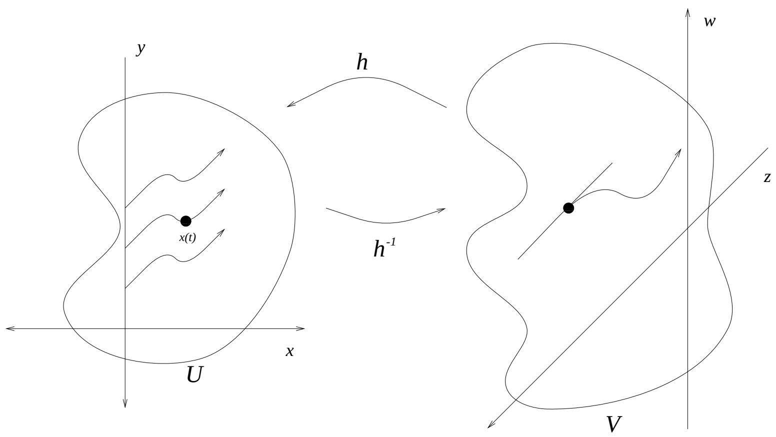

FIGURE 3. The map h.

Given $h\colon V \to U$ (i.e. new variables to old variables):

1. Move vector field: Define $g\colon V \to \mathbf{R}^n$ by

$$\begin{aligned} g(z) &= (Dh^{-1})\big|_{h(z)} f(h(z)) \\ &= \left((Dh)\big|_{h(z)} \right)^{-1} f(h(z)) \end{aligned}$$

Note:

$$D(h^{-1})\big|_{h(z)} = Dh\big|_z^{\;-1}$$

2. Move the solution curves. Given $z_0 \in V$, we have $x_0 = h(z_0)$. Use that as initial condition to obtain solution $x(t)$. Then define $z(t) = h^{-1}(x(t))$.

   Claim: These are the same operation.

   Proof:

$$\begin{aligned}\frac{dz}{dt} &= Dh^{-1}(\frac{dx}{dt})\\ &= Dh^{-1}(f(x(t)))\\ &= Dh^{-1}(f(h(z(t))))\end{aligned}$$

   So, $dz/dt = g(z)$. □

Definition 3.1. *Given (local) flows*

$$\begin{aligned}\phi &\colon \mathbf{R} \times U \to U\\ \psi &\colon \mathbf{R} \times V \to V\end{aligned}$$

we say ϕ,ψ are ***conjugate*** *if there exists a homeomorphism $h\colon V \to U$ such that*

$$h(\psi(t, z_0)) = \phi(h(t, z_0)) \tag{13}$$

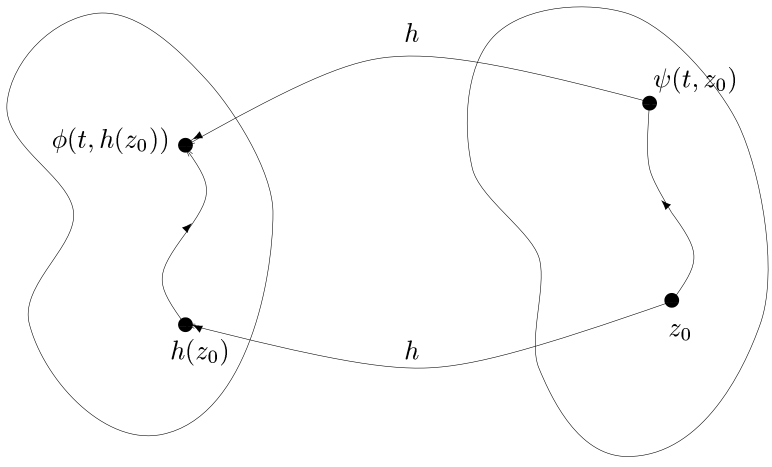

Figure 4. A conjugacy.

If h is not a diffeomorphism, we can still use as a conjugacy for flows, but we can't move the vector field.

Remark: Conjugacy is a very strong notion of "the same".

Example: $X \in \mathbf{R}^n$, $\dot{X} = AX$. Do a linear change of variables, $X = PY$, where P is an $n \times n$ matrix. Then $h(Y) = PY$, so $Dh\, x|_= P$. So, in the new variables,

$$\dot{Y} = P\dot{X} = PAX = PAP^{-1}Y$$

Example:

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues are $1 \pm \sqrt{2}$, so there is P s.t. $PAP^{-1} = \begin{pmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{pmatrix}$. So, let $X = PY$, then

$$\dot{Y} = \begin{pmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{pmatrix} Y$$

where

$$\begin{aligned} Y &= \begin{pmatrix} z \\ w \end{pmatrix}, \text{ so} \\ \dot{z} &= (1+\sqrt{2})z \text{ and} \\ \dot{w} &= (1-\sqrt{2})w \end{aligned}$$

These are conjugate linear systems.

Another change of variables is change in time variable. Start with $f \colon U \to \mathbf{R}^n$, with a solution (local) flow ϕ. Pick $x_0 \in U$, let $x(t)$ be the solution through x_0. Change to new time variable $\mathcal{S}(t)$, so new time is a function of old time. Let $\mathcal{T}(s) = t$, so that $\mathcal{S}^{-1} = \mathcal{T}$, and old time is a function of new time.

So what differential equation is $x(\mathcal{T}(s))$ a solution of?

4. Change of time variables(Sept. 11)

Start with a vector field $f \colon U \to \mathbf{R}^n$, $x_0 \in \mathbf{R}^n$. Change time variables, get new time variable s, $s = \mathcal{S}(t)$, and $t = \mathcal{T}(s)$.

For what differential equation is $x(\mathcal{T}(s))$ the solution?

Differentiate:

$$\begin{aligned} \frac{d}{ds} x(\mathcal{T}(s)) &= \dot{x}(\mathcal{T}(s)) \cdot \frac{dt}{ds} \\ &= f(x(\mathcal{T}(s))) \cdot \left.\frac{dt}{ds}\right|_s \end{aligned}$$

Effect of changing to a new time is only on the speed, not the direction.

Start with $f \colon U \to \mathbf{R}^n$, all as above. Suppose we have $\sigma \colon U \to \mathbf{R}^+$, smooth.

Then we can make a new vector field

$$g(x) = \sigma(x) f(x)$$

What are the solution curves of $\dot{x} = g(x)$? Start with an IC x_0. Let $x(t)$ be the solution of $\frac{dx}{dt} = f(x)$ with $x(0) = x_0$. The goal is to change $x(t)$ into a solution of $\frac{dx}{ds} = g(x)$ by defining a new time variable.

Suppose $\mathcal{T}(s)$ is a change of time variable such that

$$\frac{dx(\mathcal{T}(s))}{ds} = g(x(\mathcal{T}(s)))$$

But then

$$\begin{aligned} f(x(\mathcal{T}(s))) &= \left.\frac{dx}{dt}\right|_{\mathcal{T}(s)} \left.\frac{dt}{ds}\right|_s \\ &= \sigma(x(\mathcal{T}(s))) \cdot f(x(\mathcal{T}(s))) \end{aligned}$$

So, we need

$$\left.\frac{dt}{ds}\right|_s = \sigma(x(\mathcal{T}(s)))$$

If there exists such a $\mathcal{T}(s)$ then we can change from one system to another just by changing time variable. But the equation

$$\left.\frac{dt}{ds}\right|_s = \sigma(x(\mathcal{T}(s)))$$

is just a 1-dimensional ODE where we know σ and x. By the Existence-Uniqueness Theorem, there is a unique solution for each initial condition.

A nice feature is that if you change speeds (lengths) of the vector field, you don't change the phase portrait.

For example, the system

$$\begin{cases} \dot{x} = -x \\ \dot{y} = 2y \end{cases}$$

and the system

$$\begin{cases} \dot{x} = -x(x^2 + y^2 + 1) \\ \dot{y} = 2y(x^2 + y^2 + 1) \end{cases}$$

have exactly the same phase portraits.

Another notion of when flows are "the same":

DEFINITION 4.1. *Two flows $\phi\colon U \times \mathbf{R} \to U$, $\psi\colon V \times \mathbf{R} \to V$ are said to be* ***topologically equivalent*** *if there exists a homeomorphism $h\colon V \to U$ such that, $\forall y_0 \in V$,*

$$h\{\psi(t, y_0), t \in \mathbf{R}\} = \{\phi(t, h(y_0)), t \in \mathbf{R}\} \tag{14}$$

where orientation of increasing time is also preserved.

Example: Zharkovskii's model of glider flight. (Theory of Oscillators, Andronov)

Velocity:

$$v = v(\cos\theta, \sin\theta)$$
$$\dot{v} = \dot{v}(\cos\theta, \sin\theta) + v\dot{\theta}(\cos\theta, \sin\theta)$$

The model is

$$m\frac{dv}{dt} = -mg\sin\theta - \frac{1}{2}\rho F C_x v^2 \tag{15}$$
$$mv\frac{d\theta}{dt} = -mg\cos\theta + \frac{1}{2}\rho F C_y v^2 \tag{16}$$

where m is mass, g the acceleration due to gravity, ρ the density of air, F the area of the wing, C_x the resistance to motion per unit area of wing, and C_y the lift per unit area of the wing.

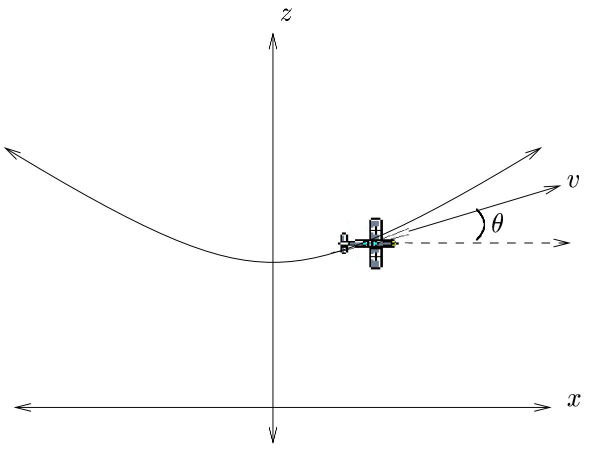

FIGURE 5. Here the glider is moving with speed v and angle θ.

So we get

$$\frac{dv}{dt} = -g\sin\theta - \frac{1}{2}\frac{\rho F C_x}{m}v^2$$
$$\frac{d\theta}{dt} = \frac{1}{v}\left(-g\cos\theta + \frac{1}{2}\frac{\rho F C_y}{m}v^2\right)$$

What are the essential parameters for the qualitative behavior? We have control of the units of distance and time. Use a new variable y for v, such that $v = ky$, with k constant.

$$\frac{dy}{dt} = -\frac{g}{k}\sin\theta - \frac{k}{2}\frac{\rho F C_x}{m}y^2$$
$$\frac{d\theta}{dt} = \frac{1}{y}\left(-\frac{g}{k}\cos\theta + \frac{k}{2}\frac{\rho F C_y}{m}y^2\right)$$

Use a new time variable τ with $\alpha\tau = t$.

$$\frac{dy}{d\tau} = \frac{dy}{dt}\frac{dt}{d\tau} = \alpha\frac{dy}{dt}$$
$$\frac{d\theta}{dt} = \alpha\frac{d\theta}{dt}$$

i.e. multiply old vector field by α to get:

$$\frac{dy}{d\tau} = -\frac{\alpha g}{k}\sin\theta - \frac{\alpha k}{2}\frac{\rho F C_x}{m}y^2$$
$$\frac{d\theta}{d\tau} = \frac{1}{y}\left(-\frac{\alpha g}{k}\cos\theta + \frac{\alpha k}{2}\frac{\rho F C_y}{m}y^2\right)$$

Pick $k = g\alpha$, and $\alpha k = \dfrac{2m}{\rho F C_y}$.
Thus

$$\alpha = \sqrt{\frac{2m}{g\rho F C_y}}$$
$$k = \sqrt{\frac{2mg}{\rho F C_y}}$$

and

$$\frac{dy}{dt} = \sin\theta - \frac{C_x}{C_y}y^2$$
$$\frac{d\theta}{dt} = \frac{-\cos\theta + y^2}{y}$$

Let $a = \dfrac{C_x}{C_y} = \dfrac{\text{drag / area}}{\text{lift / area}}$.

This is the only parameter. What does this say? This says that the qualitative behavior, up to topological equivalence, depends only on C_x/C_y.

This model works for a glider moving through air, but also works for a submarine in water.

This process is called nondimensionalization. When $ky = v$, choose a k with units meters per second, and then y has no units. Choose k to be some characteristic velocity, and can use k to focus attention on different areas of v variable.

5. Example, using the 2-body problem (Sept.13)

Collision in the 2-body problem.

$$m_1\ddot{\overline{q}}_1 = \left(\frac{-Gm_1m_2}{\|\overline{q}_1 - \overline{q}_2\|^2}\frac{\overline{q}_2 - \overline{q}_1}{\|\overline{q}_2 - \overline{q}_1\|}\right)$$
$$m_2\ddot{\overline{q}}_2 = \left(\frac{+Gm_1m_2}{\|\overline{q}_1 - \overline{q}_2\|^2}\frac{\overline{q}_2 - \overline{q}_1}{\|\overline{q}_2 - \overline{q}_1\|}\right)$$

Let

$$\overline{u} = \frac{m_1\overline{q}_1 + m_2\overline{q}_2}{m_1 + m_2}$$

be the **center of mass**.

We see that $\ddot{\overline{u}} = \overline{0}$, so $\overline{u}(t) = \overline{a}t + \overline{b}$. Thus $\dot{\overline{u}} = \overline{a}$.

Conservation of momentum tells us that

$$\overline{u} = \frac{m_1\dot{\overline{q}}_1 + m_2\dot{\overline{q}}_2}{m_1 + m_2}$$

Choosing $\overline{u}$ as above, and letting $\overline{r} = \overline{q}_1 - \overline{q}_2$, think of $(\overline{u}, \overline{r})$ as new coordinates, and only consider the $\ddot{\overline{r}}$ equation.

$$\begin{aligned} m_1\dot{\overline{q}}_1 &= \overline{p}_1 \\ m_2\dot{\overline{q}}_2 &= \overline{p}_2 \end{aligned}$$

So in $\overline{u}$,$\overline{r}$ variables,

$$\begin{aligned} \dot{\overline{u}} &= \overline{v} \\ \dot{\overline{v}} &= \overline{0} \\ \dot{\overline{r}} &= \overline{s} \\ \dot{\overline{s}} = \ddot{\overline{q}}_1 - \ddot{\overline{q}}_2 &= \frac{-G\mu\overline{r}}{\|\overline{r}\|^3} \end{aligned}$$

where $\mu = m_1 + m_2$.

Remark: This is how masses of planets are determined, looking at orbits of their moons.

DEFINITION 5.1. *A* ***constant of motion*** *or* ***integral*** *for a differential equation $\dot{x} = f(x)$ is a function from the phase space to* $\mathbf{R}$*, such that $h(x(t))$ is constant for every solution.*

For

$$\begin{aligned} \dot{\overline{r}} &= \overline{s} \\ \dot{\overline{s}} &= \frac{-G\mu\overline{r}}{\|\overline{r}\|^3}, \end{aligned}$$

let $\overline{r} = (x, y)$ and $\overline{s} = (z, w)$, then

$$H(x, y, z, w) = \frac{z^2 + w^2}{2} - G\mu\sqrt{x^2 + y^2} \tag{17}$$

is a constant of motion (energy).

Note: In fact,

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial z} \\ \dot{y} &= \frac{\partial H}{\partial w} \\ \dot{z} &= -\frac{\partial H}{\partial x} \\ \dot{w} &= -\frac{\partial H}{\partial y}, \end{aligned}$$

So, if $(x(t), y(t), z(t), w(t))$ is a solution, then

$$\frac{d}{dt}H(x,y,z,w) = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial z}\dot{z} + \frac{\partial H}{\partial w}\dot{w} = 0$$

DEFINITION 5.2. *Given a function* $H(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)\colon \mathbf{R}^{2n} \to \mathbf{R}$, *the* ***Hamiltonian system for H*** *is*

$$\begin{aligned} \dot{x_1} &= \frac{\partial H}{\partial y_1} & \dot{y_1} &= -\frac{\partial H}{\partial x_1} \\ &\vdots & &\vdots \\ \dot{x_n} &= \frac{\partial H}{\partial y_n} & \dot{y_n} &= -\frac{\partial H}{\partial x_n} \end{aligned} \tag{18}$$

Kepler's Laws:

1. Move on conic with sun at one focus.
2. Radius vector sweeps out equal areas in equal times.
3. Semimajor axis cubed is inversely proportional to the period squared.

Note: There are some collisions, i.e. some solutions cannot be continued for all time.

What happens close to collision and is the singularity removable? We want to preserve continuity as much as possible. Physically, we want to have a "bounce".

What about other force laws?

$$\ddot{\overline{r}} = \frac{-G\mu\overline{r}}{\|\overline{r}\|^{3+\epsilon}}$$

How do these orbits pass near $\overline{0}$?

$$\begin{aligned} \dot{x} &= z \\ \dot{y} &= w \\ \dot{z} &= \frac{-G\mu x}{\sqrt{x^2+y^2}^{a}} \\ \dot{w} &= \frac{-G\mu y}{\sqrt{x^2+y^2}^{a}} \end{aligned}$$

When $a = 3$ above, it is the Newtonian problem.

$$H(x,y,z,w) = \frac{z^2+w^2}{2} - \frac{G\mu}{(a-1)\sqrt{x^2+y^2}^{(a-1)}}$$

Our goal is to see what happens near $\overline{r} = \overline{0}$. First, change variables, using polar coordinates, "blowing up" the origin.

Start with

$$(x,y) = (\rho^\alpha \cos\theta, \rho^\alpha \sin\theta)$$

with α a constant.

We will also use complex notation, i.e. $x+iy = \rho^\alpha e^{i\theta}$. Also choose new velocity coordinates

$$z + iw = \rho^\beta e^{i\theta}(u+iv)$$

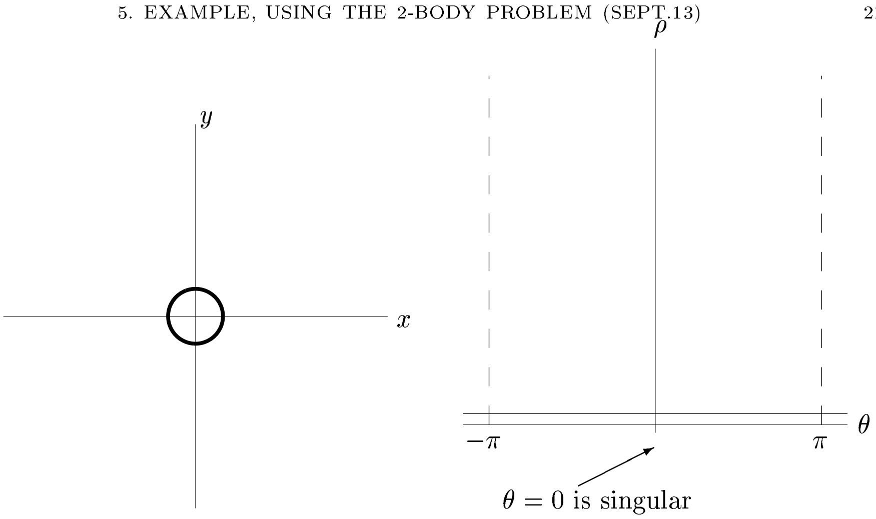

FIGURE 6. This change of coordinates is singular at $\theta = 0$.

where u is the "radial velocity", v the "angular velocity", and β a constant.

$$h\begin{pmatrix}\rho\\ \theta\\ u\\ v\end{pmatrix} = \begin{pmatrix}\rho^\alpha \sin\theta\\ \rho^\alpha\cos\theta\\ \rho^\beta(\cos\theta u - \sin\theta v)\\ \rho^\beta(\sin\theta u + \cos\theta v)\end{pmatrix} = \begin{pmatrix}x\\ y\\ z\\ w\end{pmatrix} \tag{19}$$

So we know that

$$\begin{aligned} z + iw &= \dot{x} + i\dot{y}\\ &= \frac{d}{dt}(\rho^\alpha e^{i\theta})\\ &= \alpha\rho^{(\alpha-1)}\dot{\rho}e^{i\theta} + \rho^\alpha i\dot{\theta}e^{i\theta}\end{aligned}$$

and

$$\rho^\beta e^{i\theta}(u + iv) = \alpha\rho^{\alpha-1}\dot{\rho}e^{i\theta} + \rho^\alpha i\dot{\theta}e^{i\theta}$$

so we get, finally

$$\begin{aligned} \dot{\rho} &= \frac{1}{\alpha}\rho^{\beta-\alpha+1}u\\ \dot{\theta} &= \rho^{\beta-\alpha}v.\end{aligned}$$

+

6. McGehee Collision Manifold (Sept. 16)

Recall that we have

$$\begin{aligned}\dot{x} &= z\\ \dot{y} &= w\\ \dot{z} &= \frac{-G\mu x}{\sqrt{x^2+y^2}^{a+1}}\\ \dot{w} &= \frac{-G\mu y}{\sqrt{x^2+y^2}^{a+1}}\end{aligned}$$

where $a = 2$ is the "inverse-square" Newtonian problem.

To look for a new collision, we set

$$\begin{aligned}(x, y) &= (\rho^\alpha \cos\theta, \rho^\alpha \sin\theta)\\ x + iy &= \rho^\alpha e^{i\theta}\end{aligned}$$

and

$$z + iw = \rho^\beta e^{i\theta}(u + iv)$$

and we can choose α, β below.

We ended up with

$$\begin{aligned}\dot{\rho} &= \frac{1}{\alpha}\rho^{\beta-\alpha+1}u\\ \dot{\theta} &= \rho^{\beta-\alpha}v\end{aligned}$$

To get u, v, we use:

$$\begin{aligned}\dot{z} + i\dot{w} = \beta\rho^{\beta-1}\dot{\rho}e^{i\theta}(u+iv)\\ + \rho^\beta i e^{i\theta}\theta(u+iv)\\ + \rho^\beta e^{i\theta}(\dot{u}+i\dot{v})\end{aligned}$$

and

$$\begin{aligned}\dot{z} + i\dot{w} &= -G\mu\left(\frac{x}{\sqrt{x^2+y^2}^{a+1}} + i\frac{y}{\sqrt{x^2+y^2}^{a+1}}\right)\\ &= -G\mu\frac{\rho^\alpha e^{i\theta}}{(\rho^\alpha)^{a+1}}\\ &= -G\mu\rho^{-\alpha a}e^{i\theta}\end{aligned}$$

We can solve for u and v, and after a lot of calculation, get:

$$\begin{aligned}\dot{u} &= -G\mu\rho^{-\beta-\alpha a} - \frac{\beta}{\alpha}\rho^{\beta-\alpha}u^2 + \rho^{\beta-\alpha}v^2\\ \dot{v} &= (-\frac{\beta}{\alpha}\rho^{\beta-\alpha} - \rho^{\beta-\alpha})uv.\end{aligned}$$

Now choose α, β, with the condition that $-\beta - \alpha a = \beta - \alpha$. If we also pick α so that $\beta - \alpha = 0$, then $\alpha = 0$, which doesn't work (since that would give $x + iy = \rho^0 e^{i\theta}$).

Let $\beta - \alpha = -1$, and then solve for α. In the case that $a = 2$, we get $\alpha = 2/3$, $\beta = 1/3$.

$$\begin{aligned}
\dot{\rho} &= \frac{3}{2}u \\
\dot{\theta} &= \rho^{-1}v \\
\dot{u} &= \rho^{-1}(-G\mu + \frac{1}{2}u^2 + v^2) \\
\dot{v} &= \rho^{-1}(-\frac{1}{2}uv)
\end{aligned}$$

Vector field doesn't exist at $\rho = 0$. But we can do a change of time coordinate, multiplying the vector field by ρ. This corresponds to a new time τ where $dt/d\tau = \rho$. So out new clock "runs slow" at $\rho = 0$.

Thus:

$$\begin{aligned}
\frac{d\rho}{d\tau} = \rho' &= \frac{3}{2}\rho u \\
\theta' &= v \\
u' &= -G\mu + \frac{1}{2}u^2 + v^2 \\
v' &= -\frac{1}{2}uv
\end{aligned}$$

What do we have?

1. We have removed the singularity at $\rho = 0$, and extended the vector field to $\rho = 0$.
2. Also, the system s: There are no ρ's in θ', u', v', and no θ's or ρ's in u' or v'.

Recall the energy in new variables:

$$\begin{aligned}
H(\rho, \theta, u, v) &= \rho^{2\beta}\frac{u^2 + v^2}{2} - \frac{G\mu}{(a-1)(\rho^\alpha)^{\alpha-1}} \\
x + iy &= \rho^\alpha e^{i\theta} = \rho^{2/3}e^{i\theta} \\
z + iw &= \rho^\beta e^{i\theta}(u + iv) = \rho^{-1/3}e^{i\theta}(u + iv) \\
H &= \rho^{-2/3}\frac{u^2 + v^2}{2} - \frac{G\mu}{\rho^{2/3}} \\
H(\rho, \theta, u, v) &= \rho^{-2/3}\left(\frac{u^2 + v^2}{2} - G\mu\right)
\end{aligned}$$

How do we use this?

What happens when $\rho = 0$?

$$\rho' = 0$$
$$\theta' = v$$
$$u' = -G\mu + \frac{1}{2}u^2 + v^2$$
$$v' = -\frac{1}{2}uv$$

So if $\rho(0) = 0$ then $\rho(t) = 0$ for all t.
What this means is that we have added a boundary to the phase space.

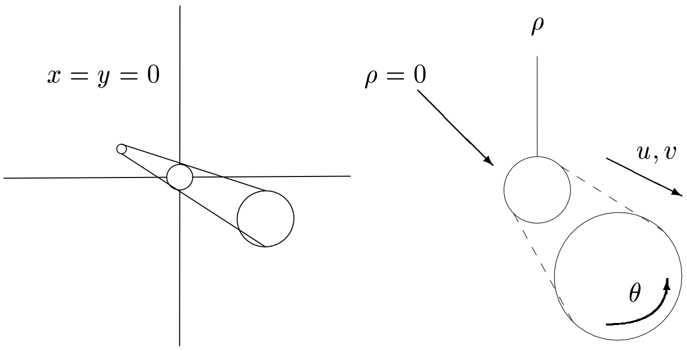

FIGURE 7. "Blowing up" the origin

If we fix $H(\rho, \theta, u, v) = h$, (look at solutions with energy h), we get

$$h = \rho^{-2/3}\left(\frac{u^2 + v^2}{2} - G\mu\right),$$
$$\rho^{2/3}h = \left(\frac{u^2 + v^2}{2}G\mu\right).$$

When $\rho = 0$ we have

$$\frac{u^2 + v^2}{2} - G\mu = 0.$$

We can say 3 things:

1. The behavior of the new vector field on new boundary $\rho = 0$ is independent of h.
2. We can restrict attention to $\rho = 0$,(θ, u, v) with $u^2 + v^2 = 2G\mu$, i.e. a circle in u,v-space.
3. Note that when $\rho = 0$, $u' = v^2/2$.

So look at the phase space $u^2 + v^2 = 2G\mu$, for $\theta \in [0, 2\pi]$. With identifications, this gives us a torus.

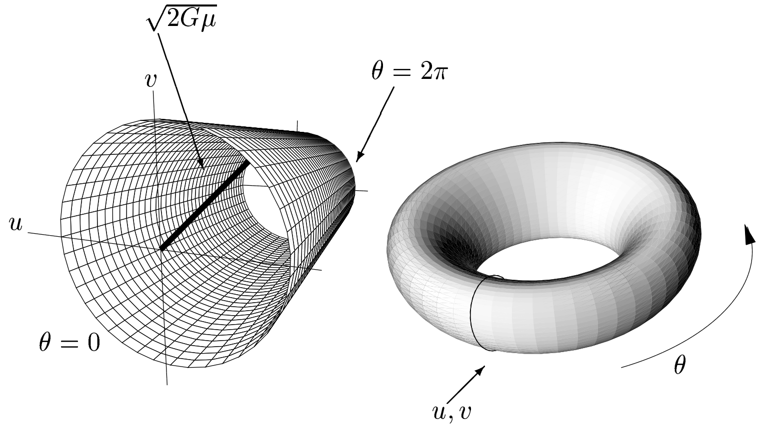

FIGURE 8. The space we restrict to, when $\rho = 0$. When we have$\rho = 0$, we know that $u^2 + v^2 = 2G\mu$, which gives us a circle in the u, v-plane, or a cylinder in u, v, θ-space. But since θ is periodic of period 2π, we can identify the "front" and the "back" of the cylinder to get a torus.

So the torus will live in (θ, u, v)-space with the restriction $u^2 + v^2 = 2G\mu$, and the vector field

$$\begin{aligned} \theta' &= v \\ u' &= \frac{v^2}{2} \\ v' &= -\frac{1}{2}uv \end{aligned}$$

Suppose I understand what happens for this flow on the torus. The original problem was to study orbits which come close to collision, i.e. come close to $\rho = 0 = x = y$. An orbit coming close to $\rho = 0$ must behave almost the same as an orbit on $\rho = 0$ because of continuity of the solution flow.

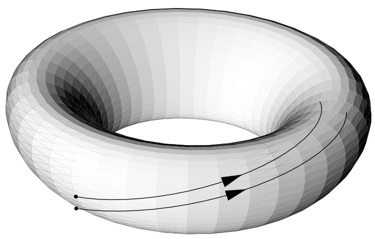

FIGURE 9. Continuity with respect to initial conditions. The orbits that start close to the orbit on the torus stay close from some time.

This torus is called the McGehee Collision Manifold. The next step will be to analyze the flow for $\rho = 0$, i.e. "solve" these equations:

$$u' = \frac{v^2}{2}$$
$$v' = -\frac{1}{2}uv.$$

7. Finishing analysis of collision manifold (Sept.18)

We have added the boundary $\rho = 0$ to our phase space. $v = 0$ gives the rest points, and if $v \neq 0$, then $u' = 0$, i.e. u is increasing.

θ increases if $v > 0$, decreases if $v < 0$. Since u is always increasing, we can make u the time variable. Since $u = u(t)$, we can think of $t = t(u)$, the inverse.

So

$$\frac{d\theta}{du} = \frac{1}{2v}$$
$$\frac{dv}{du} = -\frac{u}{v}$$

Solving, we get

$$v\,dv = -u\,du$$
$$\frac{v^2}{2} = -\frac{u^2}{2} + C$$
$$v = \pm\sqrt{C - u^2}.$$

We already know that $u^2 + v^2 = 2G\mu$, so $C = 2G\mu$. Thus

$$v = \pm\sqrt{2G\mu - u^2}$$

Plug in and solve for θ:

$$\frac{d\theta}{du} = \frac{1}{2\sqrt{2G\mu - u^2}}$$
$$\theta(u) = \int \frac{1}{2\sqrt{2G\mu - u^2}} du$$

Think about this flow. Why does it have rest points? In original problem, we know that there are solutions which go to collision as $t \to t_0$. In new variables, there must be solutions which go to $\rho = 0$ as $\tau \to \infty$.

The set of points which go to collision are the points which tend to rest points on bottom.

$$\rho' = \frac{3}{2}\rho u$$
$$u' = \frac{v^2}{2} > 0$$

A solution that goes to collision must have $u \leq 0$ for all τ. The only way is to have solution approach rest points on the bottom.

To pass close to collision means being on an orbit which is close to an orbit going to a rest point on $\rho = 0$.

Follow flow on $\rho = 0$ until near top of $\rho = 0$ then leave near an ejection orbit. There are two ways to be close to collision in some direction θ, "outside" or "inside" the sheet of orbits which go to collision.

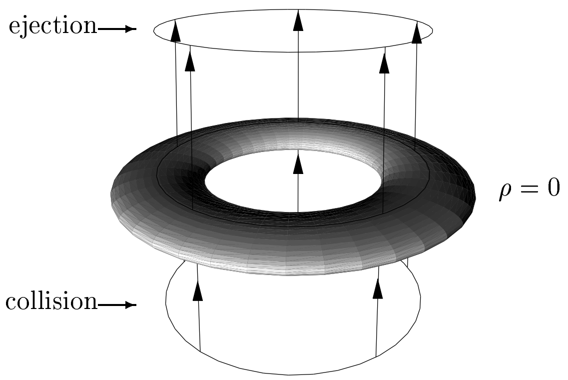

FIGURE 10. There is a circle of rest points on the to (which are unstable) and a circle on the bottom (stable). Orbits which come near the top circle are ejected, and orbits which come near the bottom circle represent near-collision orbits.

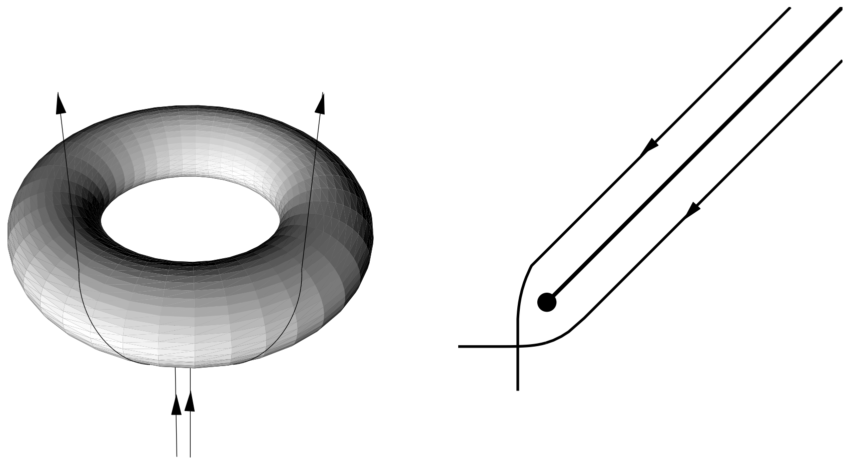

FIGURE 11. The angle at which the point-mass leaves is sensitive with respect to initial conditions. The left picture is the trajectory is phase space near the collision manifold, and the right are the corresponding paths of the masses in position space, i.e. "in reality". At this point, there is no *a priori* reason that this picture couldn't happen, even in the case of the inverse square law. The integral calculation below shows that in the case of the inverse square law, this picture could not happen, but this picture **could** happen with some other force law.

Question: When the orbit comes close to collision, in what direction does it leave collision? This is the same question as "Given an orbit on $\rho = 0$, which as

$\tau \to -\infty$ goes to $\theta(\tau) = \theta_0$. What happens to that orbit as $\tau \to \infty$?" Which rest point on the top does the orbit approach?

$$\theta(u) = 2\int_{-\sqrt{2G\mu}}^{\sqrt{2G\mu}} \frac{1}{\sqrt{2G\mu - u^2}} du = \theta(\infty) - \theta(-\infty) = 2\pi. \tag{20}$$

On $\rho = 0$, solution curves connect a rest point on the bottom to a rest point on the top which turns out to be exactly above the point on the bottom. For other force laws, this is not the case.

CHAPTER 2

The Two Big Theorems

1. Existence-Uniqueness Theorem (Sept.20)

THEOREM 1.1. *The "Big Theorem". Suppose $U \subseteq \mathbf{R}^n$, $f\colon U \to \mathbf{R}^n$ is a vector field and f is Lipschitz, i.e. there exists K such that for all x_1,$x_2 \in U$,*

$$\frac{\|f(x_1) - f(x_2)\|}{\|x_1 - x_2\|} < K. \tag{21}$$

The there exists a local flow $\phi\colon V \to U$, where $V \subseteq \mathbf{R} \times U$, and for each $x_0 \in U$, $\{t|(t, x_0) \in V\}$ is a single open interval, such that:

1. *$\forall x \in U$, $\forall t \in \mathbf{R}$ such that $(t, x) \in V$,*
$$\frac{\partial}{\partial t}\phi(t, x) = f(\phi(t, x)) \tag{22}$$
i.e. $t \mapsto \phi(t, x)$ is a solution with initial condition $x_0 = \phi(0, x)$.
2. *ϕ is continuous on V.*
3. *ϕ is **maximal** (i.e. if $x(t)$ is a solution of $\dot{x} = f(x)$ and x is defined for $t \in (-t_1, t_2)$, then $(-t_1, t_2) \times \{x(0)\} \subseteq V$.*
4. *ϕ is unique (i.e. if $x(t)$ is a solution for $t \in (-t_1, t_2)$, then $x(t) = \phi(t, x_0)$.*

Moreover, if f is $\mathcal{C}^r$, $\mathcal{C}^\infty$, $\mathcal{C}^\omega$, then ϕ is $\mathcal{C}^r$, $\mathcal{C}^\infty$, $\mathcal{C}^\omega$, respectively.

Remarks:

1. Condition 2 is called "continuity with respect to initial conditions". If we have x_1,x_2 close, and t_1,t_2 close, then $\phi(t_1, x_1)$ is close to $\phi(t_2, x_2)$. This does **not** imply that if x_1,x_2 are close, then $\phi(t_1, x_2)$ and $\phi(t_2, x_2)$ are close for all t.
2. We can extend this to cover two more cases: Given $\dot{x} = f(x, t)$, we can rewrite this as
$$\frac{dx}{ds} = f(x, t)$$
$$\frac{ds}{dt} = 1$$
This new vector field is just as smooth as f. Now, suppose f depends on parameters $\alpha \in \mathbf{R}^m$, i.e. $\dot{x} = f_\alpha(x)$. For example,
$$\dot{x} = ax + by^2$$
$$\dot{y} = cx^2 + d\cos(\alpha y)$$
has parameters a,b,c,d,α. We rewrite this as
$$\dot{x} = f_\alpha(x)$$
$$\dot{\alpha} = 0$$

which is just as smooth as f. Given $\dot{x} = f_\alpha(x)$ which is $\mathcal{C}^r$ in x and α, then $\phi_\alpha(t,x)$ is $\mathcal{C}^r$ in x,α, and t. This is known as **continuity in parameters**.

3. Much better existence theorems exist (we can even talk about f being "measurable"). However, this is the best possible result that also has uniqueness. For example:

$$f(x) = \begin{cases} \sqrt{x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We can make two solutions with $x(0) = 0$,

$$x_1(t) \equiv 0$$

$$x_2(t) = \begin{cases} 0 & t \leq 0 \\ t^2/2 & t > 0 \end{cases}$$

4. Suppose f is $\mathcal{C}^2$, then ϕ is also $\mathcal{C}^2$ and

$$\frac{\partial}{\partial t}(\phi(t,x)) = f(\phi(t,x))$$

If we differentiate with respect to x:

$$D_x \frac{\partial}{\partial t}(\phi(t,x)) = D_x f(\phi(t,x)),$$

so we get

$$\frac{\partial}{\partial t}(D_x\phi(t,x)) = D_x f\ \phi(t,x)|_(\ D_x f(\phi(t,x))).$$

For fixed t, $x \mapsto \phi(t,x)$ is called the **time t map**. $D_x\phi(t,x)$ is the $n \times n$ matrix of partials of ϕ with respect to x.

Consider the differential equation

$$\dot{X} = D_x f\ \phi(t,x_0)|_X$$

For fixed $x_0 \in U$, where $X(t)$ is an $n \times n$ matrix.

If $t = 0$, what is $D_x\phi(0,x)$? Since $x \mapsto \phi(0,x) = x$, then $D_x\phi(0,x) = I_n$.

So for fixed x_0, define the **variational equation**:

$$\dot{X} = D_x f\ \phi(t,x)|_X \tag{23}$$

$$X(0) = I_n \tag{24}$$

This is a nonautonomous linear equation on $\mathbf{R}^{n^2}$. The solution turns out to be $D_x\phi(t,x) = X(t)$.

If f is $\mathcal{C}^r$, the solution is at least $\mathcal{C}^{r-1}$.

2. Some proof of the Existence-Uniqueness Theorem (Sept.25)

We will prove a bunch of little pieces of the theorem.

THEOREM 2.1 (Cauchy's Existence Theorem). *Suppose $f\colon U \to \mathbf{R}^n$ is analytic so*

$$f(x_1, x_2, \ldots, x_n) = (f_1(x_1, x_2, \ldots, x_n), \ldots)$$

and, for $k = 1, \ldots, n$,

$$f_k(x_1, x_2, \ldots, x_n) = \sum_{l \in \mathbf{Z}^{+n}} a_{k_{l_1, l_2, \ldots, l_n}} x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$$

So, if, for example, $n = 2$, expanding about 0,

$$f(x_1, x_2) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{02}x_2^2 + a_{11}x_1x_2 + a_{20}x_1^2 + \ldots$$

and the f_k's have nonzero radius of convergence at each point.

If we expand about $x_0 = (x_{01}, \ldots, x_{0n})$,

$$f(x_1, x_2) = a_{00} + a_{10}(x_1 - x_{01}) + a_{01}(x_2 - x_{02}) + \cdots$$

Fix $x_0 \in U$. Then there exists $x(t)$ where $x\colon (-\eta, \eta) \to U$ for some $\eta > 0$ such that

1. $x(0) = x_0$
2. *$x(t)$ is a solution.*
3. *If $x(t) = (x_1(t), \ldots, x_n(t))$, then*

$$x_k(t) = \sum_{l=0}^{\infty} \alpha_{kl} t^l$$

with radius of convergence η.

Idea of Proof. This comes from Siegel and Moser's Celestial Mechanics.

Assume f is as above. Fix $x_0 \in U$, then there is an $r > 0$, M such that for all k,

$$|f_k(x)| < M \text{ for } |x - x_0| < r$$

This gives a "maximum speed" for the solution. Fix $\eta = \dfrac{r}{(n+1)M}$. This is how long (at least) we expect solution to exist.

Look for $x(t)$ solution, $x(0) = x_0$, and $\|x(t) - x_0\| < r$ for $t \in (-\eta, \eta)$.

The steps in the proof:

1. Change variables so that $x_0 = 0$, $r = M = 1$. (Can rescale x to get $r = 1$, and rescale time to get $M = 1$.)
2. Solve formally: Write for $k = 1, \ldots, n$,

$$x_k(t) = \sum_{m=0}^{\infty} a_k m t^m$$

Plug this into $\dot{x} = f(x)$, solve for α's. For example:

$$\begin{aligned}
\dot{x_1}(t) &= \sum_{m=1}^{\infty} m\alpha_{1m} t^{m-1} \\
&= f_1(x_1, x_2, \ldots, x_n) \\
&= \sum_{l \in \mathbf{Z}^{+n}} a_{1_{l_1, l_2, \ldots, l_n}} x_1^{l_1} \ldots x_n^{l_n} \\
&= \sum_{l \in \mathbf{Z}^{+n}} a_{1_{l_1, l_2, \ldots, l_n}} \left(\sum_{m=0}^{\infty} \alpha_{1m} t^m \right)^{l_1} \cdots \left(\sum_{m=0}^{\infty} \alpha_{nm} t^m \right)^{l_n}
\end{aligned}$$

Equate coefficients and solve for α's:

$$\alpha_{10} = \alpha_{20} = \cdots = \alpha_{n0} = 0$$
$$\alpha_{11}(\text{constant on left}) = a_{100\ldots0}(\text{constant on right})$$
$$\alpha_{12} = \alpha_{11}a_{110\ldots0} + \alpha_{21}a_{101\ldots0} + \ldots.$$

In general, we find that each α_{km} is a polynomial in $a_{k_{l_1,l_2,\ldots,l_n}}$ where $l_1, l_2, \ldots, l_n < m$.

3. We have a formal solution by solving for α_{km}'s in terms of a's. Does this power series for $x_k(t)$ converge? (If yes, then it must be the solution.)

   To show that $x_k(t)$'s converge use "method of majorants". This is essentially the Comparison Test – we find a power series that converges and that has coefficients bigger than the $x_k(t)$'s.

   To build the new power series (the majorant), look at
$$\dot{y} = g(y) = \sum b_{1_{l_1,l_2,\ldots,l_n}} y_1^{l_1} \ldots y_n^{l_n}$$
   with $\left|a_{k_{l_1,l_2,\ldots,l_n}}\right| \leq b_{k_{l_1,l_2,\ldots,l_n}}$. Then claim if $y_k(t) = \sum \beta_{km}t^m$ is a solution with $y(0) = 0$ then $\forall k, m, |\alpha_{km}| \leq \beta_{km}$.

   Note: The equations for the β's are the same as the equations for the α's with a's replaced with b's. The equation for α_{kn} is a polynomial in a's and has all positive coeffeicients. So replacing a's with bigger b's will make β's bigger than the α's.

4. Find a nice g whose solution we know. A fact from complex analysis: Because $|f_k| < 1(= M)$ on ball of radius $1(= r)$, we know
$$\left|a_{k_{l_1,l_2,\ldots,l_n}}\right| \leq 1.$$
   So let all b_k's be 1. Solve
$$\dot{y} = g(y) = \sum_{l_1,l_2,\ldots,l_n} y_1^{l_1} y_2^{l_2} \ldots y_n^{l_n} \ldots$$
   Note: Each component of $g(y)$ is the same.

$$\begin{aligned}\sum_{l_1,l_2,\ldots,l_n} y_1^{l_1} y_2^{l_2} \ldots y_n^{l_n} &= 1 + y_1 + y_2 + \cdots + y_n + y_1^2 + y_1y_2 + y_1y_3 + \cdots \\ &= \prod_{r=1}^{n}(1 - y_r)^{-1}\end{aligned} \tag{25}$$

   since $1 + y + y^2 + \cdots + y^n + \cdots = \dfrac{1}{1-y}$.

   So
$$\dot{y_1} = \prod_{r=1}^{n}(1 - y_r)^{-1}$$
   with I.C. $y(0) = 0$.

   Solution must have $y_1(t) = y_2(t) = \cdots$, so
$$\dot{y}(t) = (1 - y)^{-n}$$
$$y(0) = 0$$

$$y_k(t)y(t) = 1 - (1-(n+1)t)^{\frac{1}{n+1}}$$

and each $y_k(t)$ is a convergent power series for $t < \dfrac{1}{n+1}$, with $|y| < 1$. So the $x_k(t)$'s converge in a least the same size ball.

3. Some more proof of the Existence-Uniqueness Theorem (Sept.27)

Example:

$$\begin{cases} \dfrac{dx}{dt} &= a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 \\ \dfrac{dy}{dt} &= b_1 x + b_2 y \end{cases}$$

Assume $x(t) = \sum_0^\infty \alpha_m t^m$, $y(t) = \sum_0^\infty \beta_m t^m$, with initial condition $\alpha_0 = \beta_0 = 0$.

Plug in:

$$\begin{aligned} \sum m a_m t^{m-1} &= a_1 + a_2 \sum \alpha_m t^m + a_3 \sum \beta_m t^m \\ &\quad + a_4 \left(\sum \alpha_m t^m\right)^2 + a_5 \left(\sum \alpha_m t^m\right)\left(\sum \beta_m t^m\right) + a_6 \left(\sum \beta_m t^m\right)^2 \end{aligned}$$

$$\sum m\beta_m t^{m-1} = b_1 \sum \alpha_m t^m + b_2 \sum \beta_m t^m$$

So we need to equate coefficients, and we find that the α_m's and the β_m's are polynomials in a's and b's with positive coefficients. So increasing a's and b's in absolute value increases the α's and the β's.

For now, we assume that we are given $F\colon U \to \mathbf{R}^n$ Lipschitz with constant L, i.e.

$$\frac{\|f(x_1) - f(x_2)\|}{\|x_1 - x_2\|} \leq L.$$

To attack uniqueness problem, look at "separation problem". Suppose we are given two initial conditions $x_1, x_2 \in U$. Suppose we have two solutions $x_1(t)$, $x_2(t)$, with $x_i(0) = x_i$. Can we get an estimate on the distance $\|x_1(t) - x_2(t)\|$?

Only thing we know is that $\dot{x}_1 = f(x_1(t))$, so

$$\left\| \frac{d}{dt}(x_1(t) - x_2(t)) \right\| = \|f(x_1(t)) - f(x_2(t))\| \tag{26}$$

$$= L\,\|x_1(t) - x_2(t)\| \tag{27}$$

Let $z = \|x_1(t) - x_2(t)\|$. We **sort of** have that $\dot{z} \leq Lz$.

(We don't exactly because $\left\| \dfrac{d}{dt}(x_1(t) - x_2(t)) \right\| \neq \dfrac{d}{dt}\, \|(x_1(t) - x_2(t))\|$.

The worst case would be $\dot{z} = Lz$, i.e. $z(t) = e^{Lt}$. This means that how fast solutions separate depends on L, or on how fast f changes from point to point.

Change this to integral equations:

$$\int \dot{x}_i(t)\, dt = \int f(x_i(\tau))\, d\tau$$

$$x_i(t) - x_i(0) = \int_0^t f(x_i(\tau))\, d\tau$$

$$x_i(t) = x_i + \int_0^t f(x_i(\tau))\, d\tau$$

Then we have

$$\|(x_1(t) - x_2(t))\| = \left\| x_1 - x_2 + \int_0^t (f(x_1(\tau)) - f(x_2(\tau)))\, d\tau \right\| \tag{28}$$

$$\leq \|x_1 - x_2\| + \left| \int_0^t \|f(x_1(\tau)) - f(x_2(\tau))\|\, d\tau \right| \tag{29}$$

$$\leq \|x_1 - x_2\| + \left| \int_0^t L \|x_1(\tau) - x_2(\tau)\|\, d\tau \right| \tag{30}$$

Theorem 3.1 (Gronwall's Inequality). *If $\mu,\ \lambda > 0$, $\phi\colon \mathbf{R} \to [0, \infty]$ is continuous and*

$$\phi(t) \leq \mu + \lambda \left| \int_0^t \phi(\tau)\, d\tau \right|$$

then

$$\phi(t) \leq \mu e^{\lambda t}.$$

The proof is left as an exercise.

So this lemma gives us that

$$\|x_1(t) - x_2(t)\| \leq \|x_1 - x_2\|\, e^{Lt}$$

Corollary 3.1 (Uniqueness.). *Let $x_1(t)$, $x_2(t)$ be solutions with $x_1(0) = x_2(0)$. Then $\|x_1(t) - x_2(t)\| \leq 0$, i.e. $x_1(t) = x_2(t)$ for all t.*

Corollary 3.2. *If $\phi\colon \mathbf{R} \times U \to U$ is the local flow solution for $\dot{x} = f(x)$ then ϕ is continuous in x.*

Proof: Fix t. Then

$$\|\phi_t(x_1) - \phi_t(x_2)\| \leq e^{Lt}\, \|x_1 - x_2\|$$

□

4. Picard iteration (Sept. 30)

Idea. Start with a candidate for a solution and improve it iteratively, i.e. give a process for improving approximate solutions.

We know that $x(t)$ is a solution iff

$$x(t) = x_0 + \int_0^t f(x(\tau))\, d\tau$$

So given $\gamma_0 \colon \mathbf{R} \to U$, we can check to see if it is a solution to see if it satisfies

$$\gamma_0(t) = x_0 + \int_0^t f(\gamma_0(\tau))\, d\tau$$

If γ_0 is not a solution, we might hope to get a closer approximation $\gamma_1(t)$ by

$$\gamma_1(t) = x_0 + \int_0^t f(\gamma_0(\tau))\, d\tau$$

So we define an operator

$$\Gamma\colon \gamma_0 \mapsto \gamma_1$$

where

$$[\Gamma(\gamma_0)](t) = x_0 + \int_0^t f(\gamma_0(t))\, d\tau$$

A solution would be γ^* such that

$$\Gamma(\gamma^*) = \gamma^*$$

so γ^* would be a fixed point of Γ. We hope that there is only one fixed point, and $\Gamma^n(\gamma_0) \to \gamma^*$.

So we assume that $f\colon U \to \mathbf{R}^n$, f is Lipschitz with constant L. We must set up a domain for Γ, and show that Γ has fixed points.

Note: $\gamma\colon [-\eta, \eta] \to U$ for some $\eta > 0$.

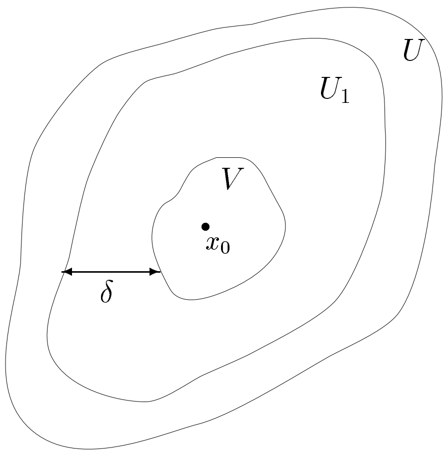

FIGURE 1. We choose δ so that $\gamma(t)$ cannot travel outside of U_1.

We want that $\Gamma(\gamma)\colon [-\eta, \eta] \to U$. We must assume that $\Gamma(\gamma)$ stays in U. Fix $U_1 \subseteq U$, compact, containing x_0. Now, let $K = \sup_{x \in U_1} \|f(x)\|$. Choose V contained in the interior of U_1, V compact, with x_0 in the interior of V, and fix $\delta < 0$ s.t. $\|x - y\| > \delta$ whenever $x \in V$, $y \in U \setminus U_1$.

Let $\eta = \delta/2K$, so if $\gamma(t)$ has speed less than K it can't travel from x_0 to outside of U_1 in time less than η.

Let $C([-\eta, \eta], U)$ be the set of continuous functions from $[-\eta, \eta]$ to U. If $\gamma_1, \gamma_2 \in C([-\eta, \eta], U)$, then

$$\|\gamma_1 - \gamma_2\| = \sup_{t \in [-\eta, \eta]} \|\gamma_1(t) - \gamma_2(t)\|\,.$$

So we know that $C([-\eta, \eta], U)$ is complete with this metric, i.e. if $\{\gamma_n\}$ is Cauchy with respect to this norm, then $\lim \gamma_n$ exists and is in $C([-\eta, \eta], U)$.

So, we really need to show that if $\gamma(0) = x_0$ and $\gamma \in C([-\eta,\eta],U)$, then $\Gamma(\gamma) \in C([-\eta,\eta],U)$.

1. $\Gamma(\gamma)$ is continuous.

$$\begin{aligned}
\|\Gamma(\gamma)(t) - x_0\| &= \left\| x_0 + \int_0^t f(\gamma(\tau))\,d\tau - x_0 \right\| \\
&\leq \left| \int_0^t \|f(\gamma(\tau))\|\,d\tau \right| \\
&\leq \left| \int_0^t \kappa\,d\tau \right| \\
&= \kappa\,|t|
\end{aligned}$$

If $t \in [-\eta,\eta]$, then

$$\|\Gamma(\gamma)(t) - x_0\| < K\eta < K(\frac{\delta}{2K}) < \delta/2$$

which means that

$$\Gamma(\gamma)(t) \in U_1 \text{ for all } t \in [-\eta,\eta].$$

Let $C([-\eta,\eta],U,x_0) = \{\gamma \in C([-\eta,\eta],U) \mid \gamma(0) = x_0\}$.
Then we see that $\Gamma\colon C([-\eta,\eta],U,x_0) \to C([-\eta,\eta],U,x_0)$.

2. Γ is a contraction.

Pick $\gamma_1,\gamma_2 \in C([-\eta,\eta],U,x_0)$. Look at

$$\begin{aligned}
\|\Gamma(\gamma_1) - \Gamma(\gamma_2)\| &= \sup_{t\in[-\eta,\eta]} \|[\Gamma(\gamma_1)(t) - \Gamma(\gamma_2)(t)]\| \\
&= \sup_{t\in[-\eta,\eta]} \left\| \int_0^t f(\gamma_1(t))\,d\tau - \int_0^t f(\gamma_2(t))\,d\tau \right\| \\
&\leq \sup_{t\in[-\eta,\eta]} \left\| \int_0^t \|f(\gamma_1(\tau)) - f(\gamma_2(\tau))\|\,d\tau \right\| \\
&\leq \sup_{t\in[-\eta,\eta]} \left\| \int_0^t L\,\|\gamma_1(\tau) - \gamma_2(\tau)\|\,d\tau \right\| \\
&\leq \sup_{t\in[-\eta,\eta]} \left| \int_0^t L\,\|\gamma_1 - \gamma_2\|\,d\tau \right| \\
&= \sup_{t\in[-\eta,\eta]} |t|\,L\,\|\gamma_1 - \gamma_2\| \\
&\leq \eta L\,\|\gamma_1 - \gamma_2\|
\end{aligned}$$

We have to have $\eta L = \lambda < 1$. Make η smaller if necessary, i.e.

$$\eta < \min(\frac{1}{2L}, \frac{\delta}{2K})$$

We get that

$$\|\Gamma(\gamma_1) - \Gamma(\gamma_2)\| < \lambda\,\|\gamma_1 - \gamma_2\|$$

Thus if $\gamma_0 \in C([-\eta,\eta],U,x_0)$, we can construct $\gamma_{n+1} = \Gamma(\gamma_n)$, so that

$$\begin{aligned}\|\gamma_n - \gamma_{n+1}\| &= \|\Gamma^n(\gamma_0) - \Gamma^n(\gamma_1)\| \\ &< \lambda^n \|\gamma_0 - \gamma_1\|.\end{aligned}$$

So if $m > n$,

$$\begin{aligned}\|\gamma_m - \gamma_n\| &\leq \|\gamma_n - \gamma_{n+1}\| + \cdots + \|\gamma_{m-1} - \gamma_m\| \\ &\leq (1 + \lambda + \lambda^2 + \cdots + \lambda^{m-n}) \|\gamma_n - \gamma_{n+1}\| \\ &= \frac{1 - \lambda^{m-n+1}}{1 - \lambda} \lambda^n \|\gamma_0 - \gamma_1\|\end{aligned}$$

So $\{\gamma_n\}$ is Cauchy, thus converges to γ^*, which is fixed because

$$\begin{aligned}\Gamma(\gamma^*) &= \Gamma(\lim_{n\to\infty} \gamma_n) \\ &= \lim_{n\to\infty} \Gamma(\gamma_n) \\ &= \lim_{n\to\infty} \gamma_{n+1} \\ &= \gamma^*\end{aligned}$$

Also, we know this fixed point is unique, so we're done.
Note: γ^* is defined on a short interval of time. Also,

$$\gamma^*(t) = x_0 + \int_0^t f(\gamma^*(\tau))\, d\tau$$

so

$$\dot{\gamma}^*(t) = f(\gamma^*(t))$$

and is obviously differentiable.

5. Invariant Sets (Oct. 2)

From now on we will study a flow $\phi\colon \mathbf{R} \times U \to U$ associated to a vector field $f\colon U \to \mathbf{R}^n$, where U is a surface in $\mathbf{R}^n$.

Ideally, we would like to find some sort of algorithm for going from x_0 to complete information about $\mathcal{O}(x_0) = \{\phi(t, x_0) \mid t \in \mathbf{R}\}$.

Since we can't do all of that, we have two choices:

1. Get a lot of info about a few orbits, or
2. Get a little info about a lot of orbits.

Definition 5.1. *Given a flow $\phi\colon \mathbf{R} \times U \to U$, an **invariant set** is a subset $I \subseteq U$ s.t. for all $x_0 \in I$, $\mathcal{O}(x_0) \subseteq I$. In other words, if $x_0 \in I$, then $\phi(t, x_0) \in I$ for all t. A **positively invariant set** is a set $J \subseteq U$ s.t. if $x_o \in J$, then $\mathcal{O}^+(x_0) \subseteq J$.*

Definition 5.2. *An **isolated invariant set** is a subset $I \subseteq U$ satisfying*

1. *I is invariant,*
2. *I is compact,*
3. *There exists $N \subseteq U$ (N compact) s.t. $I \subseteq N^o$ and I is the maximal invariant set in N. (If $J \subseteq N$, J invariant, then $J \subseteq I$.)*

*N is called an **isolating neighborhood**.*

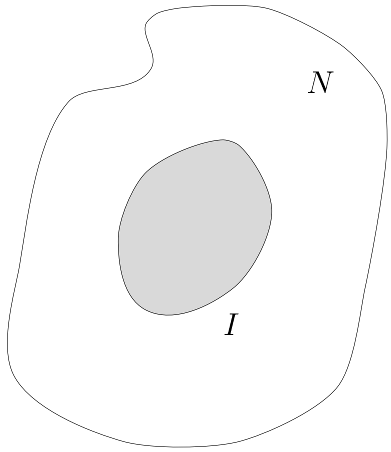

FIGURE 2. A maximal invariant set.

LEMMA 5.1. *If I is the maximal invariant set in N, and $x_0 \in N \setminus I$, then there is a t such that $\phi(t, x_0) \notin N$.*

Proof: If $\mathcal{O}(x_0) \subseteq N$ then $\mathcal{O}(x_0)$ is invariant, so $\mathcal{O}(x_0) \subseteq I$. $\square$

THEOREM 5.1. *Suppose I,J are invariant for ϕ. Then $I \cap J$, $I \cup J$, $\overline{I}$, and I^o are all invariant sets. If $x_0 \in I \cap J$, then $\mathcal{O}(x_0) \subseteq I$, $\mathcal{O}(x_0) \subseteq J$.*

Philosophy. We study flows by studying invariant sets, and there are two ways to do this:

1. What's the flow like restricted to an invariant set?
2. How are invariant sets connected?

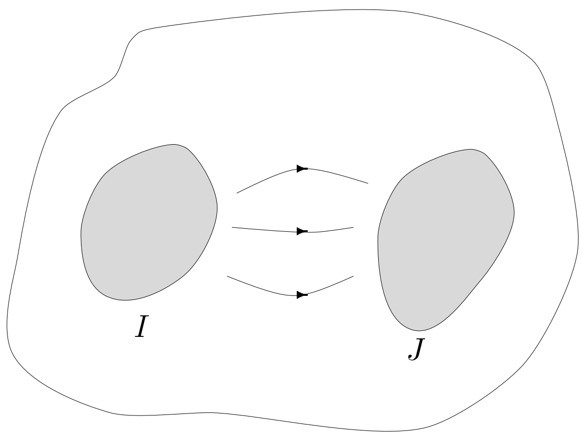

FIGURE 3. Orbits moving from one maximally invariant set to another

Are there orbits $\phi(t, x) \to J \quad (\text{ as } t \to \infty)$, or $\phi(t, x) \to I \quad (\text{ as } t \to -\infty)$?

Definition 5.3. *For ϕ a flow and x an initial point, the* ω-***limit set*** *of x is*

$$\bigcap_{t>0} \overline{\phi([t,\infty),x)},$$

where the overbar denotes closure. The α-***limit set*** *is the same thing in backwards time:*

$$\bigcap_{t>0} \overline{\phi((\infty,t],x)}.$$

The first type of invariant sets are rest points. Question 1 is easy. For question 2, we ask, what do orbits near the rest points do?

We use calculus (mostly) to study the local behavior.

Example: $\dot{x} = Ax$, $x \in \mathbf{R}^n$, A is a $n \times n$ matrix. We know that $x = 0$ is a rest point. It is isolated as a rest point, provided that $\det(A) \neq 0$ (no other rest points nearby). We have to be careful, however. For example, let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It has eigenvalues $\pm i$. $x = 0$ is not an isolated invariant set.

We will see that the condition guaranteeing that $x = 0$ is isolated is that all eigenvalues have nonzero real part.

We will get the flow

$$\phi(t,x) = e^{At}x$$

We will start to classify the linear systems:

1. Find eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (with repeats).
2. Put A into normal form, i.e. find P s.t. $P^{-1}AP = B$ where

$$B = \begin{bmatrix} \begin{bmatrix} \lambda_1 & 1 & & 0 \\ 0 & \ddots & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} a_1 & -b_1 & 1 & 0 \\ b_1 & a_1 & 0 & 1 \\ 0 & 0 & a_1 & -b_1 \\ 0 & 0 & b_1 & a_1 \end{bmatrix} & 0 \\ 0 & 0 & \ddots \end{bmatrix}$$

Essentially, put the matrix in Jordan canonical form.

3. Change variables $x = Py$, so that $\dot{y} = P^{-1}APy = By$, and if

$$B = \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix}$$

then

$$y(t) = e^{Bt} = \begin{pmatrix} e^{B_1 t} & 0 \\ 0 & \ddots \end{pmatrix} y_0.$$

4. Classify each

$$\dot{y} = e^{B_j t} y_0.$$

We will try to classify the nonlinear behavior in the same way: linearize, and try to describe the behavior near $x = 0$.

Now we can start classifying.

DEFINITION 5.4. *An isolated invariant set I is called an* ***attractor*** *if there exists an isolating neighborhood N (so I is maximal invariant set in N) such that for all $x \in N$, $\phi(t, x) \in N^o$, for all $t > 0$. Such an N is called an* ***attractor block****. An attracting fixed point is called a* ***sink****.*

LEMMA 5.2. *If N is an attractor block for I, then for all $x \in N$, $\omega(x) \subseteq I$.*

Proof: Now that $\omega(x)$ is an invariant set (exercise), so $\omega(x) \subseteq I$, since I is maximal in N, and $\omega(x) \subseteq N$. $\square$

In a weak sense, sinks are all the same, as we see by the following

THEOREM 5.2. *If ϕ,ψ are flows on $\mathbf{R}^n$ with $N \subseteq \mathbf{R}^n$ an attractor block for both ϕ and ψ with maximal invariant sets fixed points x_0 for ϕ, and y_0 for ψ, then $\phi|_N$ and $\psi|_N$ are conjugate, i.e. there is a homeomorphism $h \colon N \to N$ such that*

$$\psi(t, h(x)) = h(\phi(t, x)).$$

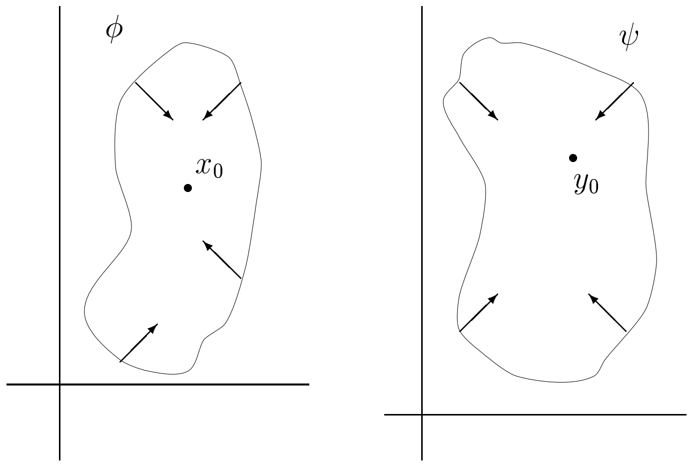

FIGURE 4. These flows are conjugate.

LEMMA 5.3. *If $x \in N$, and $x_0 \neq x$, then $\exists t \leq 0$ such that $\phi(t, x) \in \partial N$ and $\phi(s, x) \in N^o$ for $0 > s > t$.*

Proof: If not, then $\mathcal{O}_\phi(x) \subseteq N$ implies that $\mathcal{O}_\phi(x) \subseteq \{x_0\}$ because $\{x_0\}$ is maximal. $\Rightarrow\Leftarrow$ $\square$

Idea of proof of theorem.

1. Define $h|_{\partial N}$ = identity.
2. For $x \in N$, $x \neq x_0, x \in N^o$, pick first $t < 0$ such that $\phi(t, x) \in \partial N$. So define $h(x) = \psi(-t, h(\phi(t, x)))$.
3. Let $h(x_0) = y_0$.
4. Need to show that h is continuous.

6. Sinks and conjugacy (Oct. 4 and Oct. 7)

Given two flows with sinks, we know that these flows are locally conjugate near the sinks:

1. Define h on boundaries.
2. Extend h inside using flows.
3. Show continuity of h (and h^{-1}).

Note: h is only a homeomorphism, but this is the best one can hope for.

Example: Suppose that

$$\dot{x} = f(x)$$
$$\dot{y} = g(y)$$

with $f(0) = g(0) = 0$.

Suppose there is $h\colon U \to V$ where U is a neighborhood of $\overline{0}$ in y-space, and V is a neighborhood of $\overline{0}$ in x-space such that h is a homeomorphism and takes g to f.

i.e. $g(y) = \dot{y} = Dh|_x\ \dot{x} = Dh|_x\ f(x)$.

$$f(x) = f(h(y)) = \dot{x} = Dh|_y\ \dot{y} = Dh|_y\ g(y),$$

i.e. h is a smooth conjugacy between solution flow of $\dot{x} = f(x)$ and $\dot{y} = g(y)$. If h was smooth, $f(h(y)) = Dh|_y\ g(y)$, so

$$f(h(0)) = Dh|_y\ g(0) = 0.$$

So $h(0)$ is a fixed point of f, assume $h(0) = 0$. Take $f(h(y)) = Dh|_y\ g(y)$ and differentiate:

$$Df|_{h(y)} \cdot Dh|_y = D^2h\big|_y\ g(y) + Dh|_y \cdot Dg|_y$$

At $y = 0$, we get

$$\begin{aligned} Df|_0 \cdot Dh|_0 &= D^2h\big|_0\ g(0) + Dh|_0 \cdot Dg|_0 \\ Df|_0 \cdot Dh|_0 &= Dh|_0 \cdot Dg|_0 \\ Df|_0 &= Dh|_0 \cdot Dg|_0 \cdot (Dh|_0)^{-1} \end{aligned}$$

So, if $n = 1$, we get $f'(0) = g'(0)$.

If $n > 1$, then $Df|_0$ is a conjugate matrix of $Dg|_0$. So $Df|_0$ and $Dg|_0$ have the same eigenvalues, and the same dimension e-spaces (same Jordan form).

Given 2 vector fields f,g with fixed points at 0. If solution flows of f and g are smoothly conjugate, then $Df|_0$ and $Dg|_0$ are similar matrices.

Example: $\dot{x} = -x, \dot{y} = -2y$

These are certainly topologically the same, but h cannot be differentiable at 0.

So if we have $\dot{x} = f(x)$ with $f(0) = 0$ then $Df|_0$ should help in determining local structure. Recall, we can write

$$f(x) = f(0) + Df|_0\ x + \text{ higher order terms}$$

So our system $\dot{x} = f(x)$ is

$$\dot{x} = Df|_0 \, x + \text{ h.o.t.}$$

Given $\dot{x} = f(x)$, $f(0) = 0$, can you tell if 0 is a sink?

THEOREM 6.1. *If $\dot{x} = f(x)$ has 0 as a rest point and the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $Df|_0$ all have real part negative, then 0 is a sink.*

Remark: Assume that f is $\mathcal{C}^2$.

Proof: Show that there is a one-parameter family of attractor blocks around 0 which limit onto 0. (This will show that $\{0\}$ is an isolated invariant set and that 0 is a sink.)

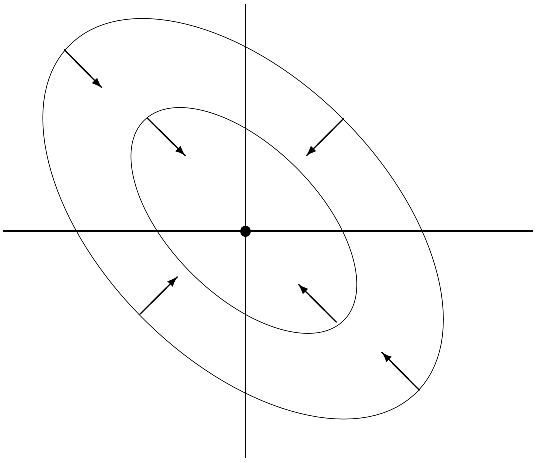

FIGURE 5. A sink

There are two big steps:

1. Compare $\dot{x} = f(x)$ to $\dot{x} = Ax$ where $A = Df|_0$.
2. Show attractor block results for $\dot{x} = Ax$.

1. Write $\dot{x} = f(x) = Ax + g(x)$ where $g(x) = f(x) - Ax$. There there is a $K > 0$ such that for all $\|x\| < 1$,

$$\|g(x)\| \le K \|x\|^2$$

We know this is true in one dimension for $\mathcal{C}^2$ functions by Taylor's Remainder Theorem. For each x_0 with $\|x_0\| = 1$ look at the map

$$s \mapsto f(sx_0) - A(sx_0)$$

and each component satisfies the 1-dimensional Taylor theorem. So

$$\|f(sx_0) - A(sx_0)\| < K_{x_0} s^2$$

K_{x_0} is determined by $\dfrac{d^2}{ds^2} f(sx_0)$, which is determined by $D^2 f$. But all second partials of f are uniformly bounded in the unit ball so we can replace K_{x_0} by K.

2. Find P such that $P^{-1}AP = B$, where B is the Jordan form of A, and let $x = Py$. Then

$$P\dot{y} = \dot{x} = f(x) = Ax + g(x) = APy + g(Py)$$

and

$$\dot{y} = By + P^{-1}(g(Py))$$

Let $g_1(y) = P^{-1}(g(Py))$.

Claim: $\|g_1(y)\| < K_1 \|y\|^2$ provided that $\|y\| < \delta$, for some fixed $\delta > 0$.

Proof:

$$\left\|P^{-1}(g(Py))\right\| \leq \left\|P^{-1}\right\| \, \|g(Py)\| \leq \left\|P^{-1}\right\| \, K \, \|Py\|^2,$$

provided that $\|Py\| < 1$. But there is a $\delta_1 > 0$ such that if $\|y\| < \delta_1$, then $\|Py\| < 1$, and

$$\|g_1(y)\| < \underbrace{\left\|P^{-1}\right\| K \|P\|}_{K_1} \|y\|^2$$

$\square$

If B were diagonal, then life would be simple. B might be of the form

$$\begin{pmatrix} B_1 & & \\ & B_2 & \\ & & \ddots \end{pmatrix}, B_1 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \text{ for example.}$$

3. Make off-diagonal terms small in B. Suppose

$$B_1 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Look for a diagonal matrix P such that

$$P^{-1}BP = \begin{pmatrix} \lambda & \epsilon & 0 \\ 0 & \lambda & \epsilon \\ 0 & 0 & \lambda \end{pmatrix}.$$

It is easy to check that

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon^2 \end{pmatrix}.$$

As an exercise, choose P for

$$B = \begin{pmatrix} a & b & 1 & 0 \\ b & a & 0 & 1 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{pmatrix}$$

and end up with P_ϵ s.t. all off-diagonal terms of $P_\epsilon^{-1}BP_\epsilon$ are ϵ.

Let $y = P_\epsilon z$. The new equation is

$$\dot{z} = \underbrace{P_\epsilon^{-1}BP_\epsilon}_{B_1} z + \underbrace{P_\epsilon^{-1}g_1(P_\epsilon(z))}_{g_2(z)}$$

There is a $\delta_2 > 0$ s.t. if $\|z\| < \delta_2$, then $\|g_2(z)\| < K_2 \|z\|^2$.

4. Check that in these coordinates, for $r > 0$, if $\|z\| \leq r$ then

$$(B_\epsilon z + g_2(z)) \cdot z < 0,$$

i.e. the vector field on the boundary points into the ball, so ball of radius r is an attractor block.

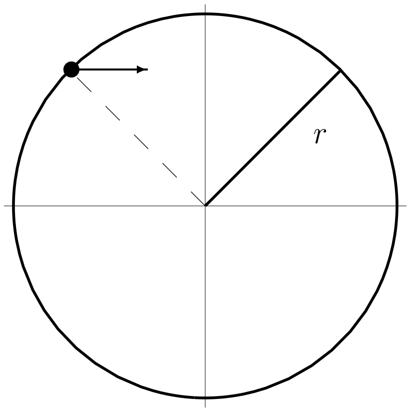

FIGURE 6. Any point on the boundary of the circle has a vector which points inside, so that this ball is an attractor block.

We want to show that $B_\epsilon z \cdot z$ is sufficiently negative so that $g_2(z) \cdot z$ can't make it positive.

Need to induct on size of Jordan blocks of B:

$$B_\epsilon = \begin{pmatrix} B_1 & & 0 \\ & B_2 & \\ 0 & & \ddots \end{pmatrix},$$

so

$$B_\epsilon z \cdot z = B_1 z_1 \cdot z_1 + B_2 z_2 \cdot z_2 + \cdots$$

So we have several cases:

1. $B_i = [\lambda_i]$, a simple real eigenvalue, then

$$B_i z_i \cdot z_i = \lambda_i z_i^2 < 0, \text{ for } \lambda_i < 0.$$

2.

$$B_i = \begin{pmatrix} \lambda & \epsilon & & \\ & \lambda & \epsilon & \\ & & \lambda & \epsilon \\ & & & \lambda \end{pmatrix}, \quad (B_i z) \cdot z = \sum_{j=1}^{m} \lambda z_j z_j + \epsilon z_{j+1} z_j$$

Now

$$\left\| \epsilon \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} z \cdot z \right\| \leq \epsilon \left\| z \right\|^2 .$$

So provided that $\epsilon < |\lambda| / 2$, then $B_1 z \cdot z < \lambda \left\| z \right\|^2 + \epsilon \left\| z \right\|^2 < 0$ because $\lambda < 0$.

3.

$$B_i = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad (B_i z) \cdot z = a(z_1^2 + z_2^2) \leq 0$$

4.

$$B_i = \begin{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} & \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} & 0 \\ & \ddots & \\ 0 & & \end{pmatrix}, \quad (B_i z) \cdot z \leq a \left\| z \right\|^2 + \epsilon \left\| z \right\|^2 .$$

which is < 0 provided that $\epsilon < |a| / 2$.

So provided that ϵ is sufficiently small (less than half the real part of the eigenvalue closest to the imaginary axis) then $B_\epsilon z \cdot z < \epsilon/2 \left\| z \right\|^2$, i.e. if the off-diagonal terms of B_ϵ are sufficiently small then

$$B_\epsilon z \cdot z \approx (\text{Diag } B_\epsilon) \cdot z < 0.$$

But we wanted

$$(B_\epsilon z + g_2(z)) \cdot z < 0$$

But note that

$$\begin{aligned} \left\| g_2(z) \cdot z \right\| &\leq K_2 \left\| z \right\|^2 \cdot \left\| z \right\| \\ &= K_2 \left\| z \right\|^3 \end{aligned}$$

provided that $\left\| z \right\| < \delta_2$.

Thus we have

$$\begin{aligned} (B_\epsilon z + g_2(z)) \cdot z &= B_\epsilon z \cdot z + g_2(z) \cdot z \\ &< -\epsilon \left\| z \right\|^2 + K_2 \left\| z \right\|^3 \\ &= \left\| z \right\|^2 (-\epsilon + K_2 \left\| z \right\|). \end{aligned}$$

So provided that $r < \epsilon / 2K_2$, we have

$$(B_\epsilon z + g_2(z)) \cdot z < 0.$$

Pull back to original coordinates:

So we get a family of attractor blocks about $x = 0$ in the original variables. Thus 0 is a sink.

$\square$

Remark: We showed that two sinks are conjugate by homeomorphism. We didn't expect the conjugacy the be smooth, because then the eigenvalues would have to be the same.

For example, if we have a system $\dot{x} = f(x)$ and $\dot{y} = g(y)$, and ϕ is a solution flow for f, and ψ is a solution flow for g, and there exists $h\colon N \to N$ such that

$$h(\psi(t,y)) = \phi(t, h(y))$$

Now, if h is $\mathcal{C}^2$, then $Df|_{x_0}$ and $Dg|_{y_0}$ are similar matrices.

Example:

$$\begin{aligned}\dot{x} &= -x\\ \dot{y} &= -2y\end{aligned}$$

Then

$$\begin{aligned}\phi(t,x) &= e^{-t}x\\ \psi(t,y) &= e^{-2t}y\end{aligned}$$

Assume that there is an $h\colon (-1,1) \to (-1,1)$ which conjugates, so that

$$h(e^{-2t}y) = e^{-t}h(y)$$

But h can't be $\mathcal{C}^1$ at 0. Suppose $h(y_0) = x_0$ so that

$$h(e^{-2t}y_0) = e^{-t}x_0$$

and

$$\sqrt{e^{-2t}y_0} = e^{-t}\sqrt{y_0} = e^{-t}kx_0$$

So $h(y) = \sqrt{y}/k$, and $h'(0)$ cannot exist.

So far, what we have is that if $\dot{x} = f(x)$ on $\mathbf{R}^n$, and $f(x_0) = 0$, then $Df|_{x_0}$ having all eigenvalues in left half-plane implies a sink, so all sinks in $\mathbf{R}^n$ are the same with respect to a continuous change of variables.

A symmetric theorem gives us that if $Df|_{x_0}$ has all eigenvalues in the right half-plane, then x_0 is a source (i.e. if we replace f by $-f$, x_0 is a sink.)

Two questions:

1. What about "mixed cases" (some eigenvalues with positive real part, some with negative)?
2. Can we do a better classification (smoother) if we take eigenvalues into account?

Definition 6.1. *Given $\dot{x} = f(x)$, f a smooth vector field on $\mathbf{R}^n$, $f(x_0) = 0$, we say that x_0 is **hyperbolic** if none of the eigenvalues of $Df|_{x_0}$ have real part $= 0$.*

We will study hyperbolic rest points, with two attacks:

1. Try to show that $\dot{x} = f(x)$ is conjugate to $\dot{y} = Ay$ near x_0 for some $A = Df|_{x_0}$,
2. Try to single out the directions in which x_0 looks like a sink or a source.

7. Preliminary to Stable/Unstable Manifold Theorem (Oct. 9)

Our goal is to understand orbits in a neighborhood of fixed points.

Given $f\colon \mathbf{R}^n \to \mathbf{R}^n$, $f(0) = 0$, we have two attacks:

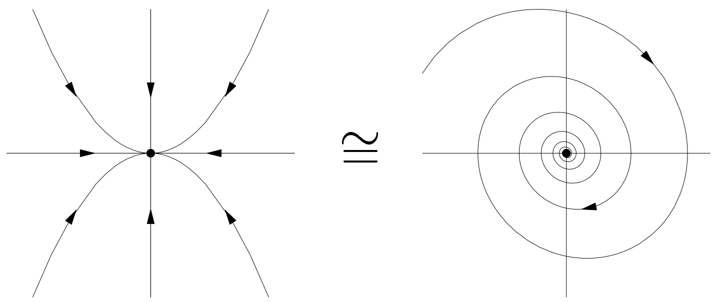

FIGURE 7. This is a continuous conjugacy, but it is obviously not a differentiable conjugacy.

1. Let $A = Df|_0$, try to find a conjugacy between solution flow of $\dot{x} = f(x)$ and solution flow of $\dot{y} = Ay$ (a local conjugacy). This is good in that it gives us info about all the orbits near 0. This is bad in that in general, we can't hope for a differentiable conjugacy (it will only be a homeo).
   (Hartman-Gröbman-Sturmberg Theorem)
2. Try to separate out the most important/interesting orbits and show that they have nice structures. Suppose $A = Df|_0$ have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$, $\mu_1, \mu_2, \ldots, \mu_u$ where the real part of the λ's are < 0, and the real part of the μ's are > 0.
   If 0 is hyperbolic, suppose A is in real Jordan form:
$$\begin{pmatrix} \mu \text{ blocks} & 0 \\ 0 & \lambda \text{ blocks} \end{pmatrix}$$

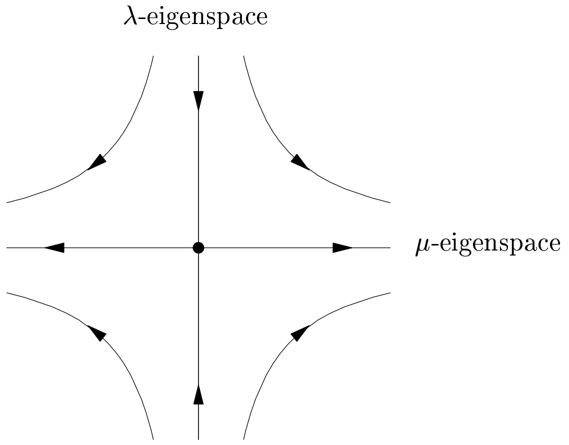

FIGURE 8. This is a saddle point; the λ's are stable and the μ's are unstable.

We know that from part 1 that near 0, the phase space of the solution to $\dot{x} = f(x)$ "looks like" this up to homeomorphism:

We want to show that the set of points whose orbits go to 0 as $t \to \infty$, or as $t \to -\infty$, are nice smooth manifolds, i.e. recover some of the structure of the linear flow by restricting solutions that go to 0 as $t \to \pm\infty$.

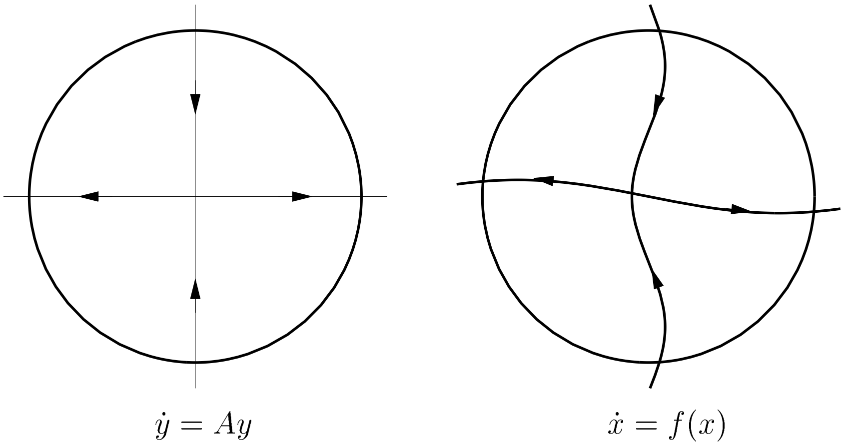

FIGURE 9. The flow will look like its linear part.

THEOREM 7.1 (Hartman-Gröbman-(part of)Sturmberg). *Let x_0 be a hyperbolic fixed point for $f\colon \mathbf{R}^n \to \mathbf{R}^n$ (where f is $\mathcal{C}^2$) vector field, with ϕ a solution flow for $\dot{x} = f(x)$. Let $A = Df|_{x_0}$, then there is a neighborhood V of x_0 and a homeomorphism $h\colon V \to V$ which locally conjugates ϕ to the solution flow for $\dot{x} = A(x - x_0)$, i.e. if $x \in V$, then*

$$\phi(t, h(x)) = h(x_0 + e^{At}(x - x_0))$$

for all t with both sides in V for time between 0 and t.

Remarks:

1. Note comparing solutions of $\dot{x} = f(x)$ to $\dot{x} = A(x - x_0)$ where $f(x) = A(x - x_0)+$ h.o.t.
2. h is only guaranteed to be a homeomorphism.

Moreover, if $n = 2$ and f is $\mathcal{C}^2$, then h can be a $\mathcal{C}^1$ diffeomorphism (Hartman). Also, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of the linear part $Df|_{x_0}$, $f \in \mathcal{C}^\infty$, and for all k, $\lambda_k \neq \sum_{j=1}^m m_j\lambda_j$ for $m_1, m_2, \ldots, m_n \in \mathbf{Z}^+ \cup \{0\}$ with $m_j \geq 2$, then we can choose h to be $\mathcal{C}^1$. The converse is not strictly true. For example, it is forbidden that A has dimension 3, with $\lambda_1 = \lambda_2 + \lambda_3$, called "resonance" in the eigenvalues.

Remark: $\mathcal{C}^1$ is good, since spiral sinks are distingushed from straight-line sinks, etc.

DEFINITION 7.1. *Suppose $X \subseteq \mathbf{R}^n$ and $X = g(\mathbf{R}^m)$ or $g(U)$, $U \subseteq \mathbf{R}^m$, such that $g\colon \mathbf{R}^m \to \mathbf{R}^n$, and*

1. *g is one-to-one, and*
2. *$Dg|_x$ ($n \times m$ matrix) has rank m at all $x \in \mathbf{R}^m$.*

*Then X is called an **immersed m-submanifold**.*

Example: $m = 1$,$n \geq 2$, then $g\colon \mathbf{R} \to \mathbf{R}^n$, if $g(\mathbf{R})$ immersed then:

1. $g(\mathbf{R})$ doesn't cross itself,
2. $g' \neq 0$ for every $x \in \mathbf{R}$, and $g(x) = (g_1(x), g_2(x))$, and $g'(x) = g_1'(x), g_2'(x))$, so g_1' and g_2' are never simultaneously 0.

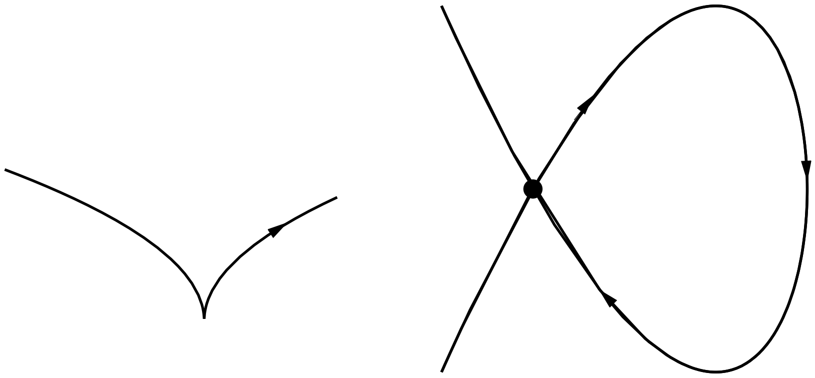

FIGURE 10. These are not allowed. The first has a cusp and the second intersects itself.

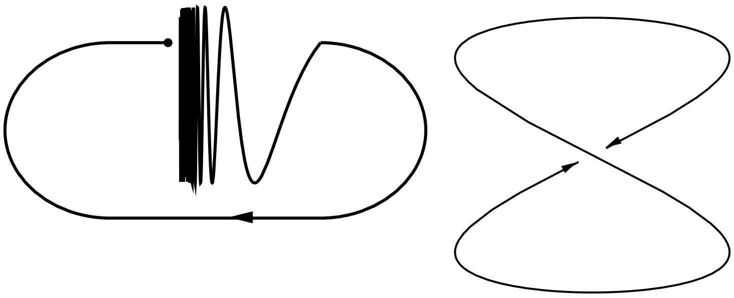

FIGURE 11. These are allowed. The manifold can limit onto itself.

We say that X is $\mathcal{C}^k$ if g if $\mathcal{C}^k$.
An alternate definition:

DEFINITION 7.2. *$X \subseteq \mathbf{R}^n$ is an immersed m-submanifold if for every $x_0 \in X$, there is a neighborhood of x_0 in $\mathbf{R}^n$ such that if V is the component of X containing x_0 in $X \cup U$ then V is the graph of a smooth function from a neighborhood of 0 in $\mathbf{R}^m$ to $\mathbf{R}^{n-m}$, i.e. we can choose coordinates in U s.t.*

$$V = \{(x_1, x_2, \dots, x_m, \eta(x_1, x_2, \dots, x_m)), \quad \textit{where } (x_1, x_2, \dots, x_m, 0, \dots, 0) \in U\}$$

and η is differentiable.

V is a graph of a smooth function, so the only possible weirdness is "global" rather than local, i.e. if you are inside X, it looks like $\mathbf{R}^m$ locally.

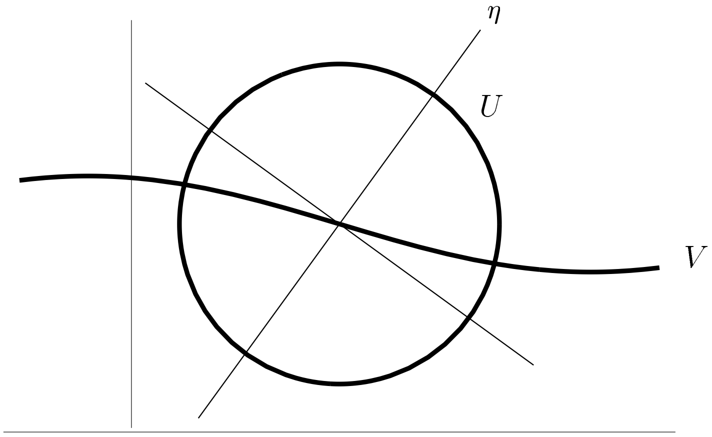

FIGURE 12. This will be a graph of a continuous function if the coordinates are chosen correctly.

8. Stable/Unstable Manifold Theorem(Oct. 15)

DEFINITION 8.1. *Let $\dot{x} = f(x)$, $f\colon \mathbf{R}^n \to \mathbf{R}^n$, a $\mathcal{C}^k$-vector field with $f(x_0) = 0$. For V a neighborhood of x_0, we define the* ***local stable set***

$$W^s(x_0, V) = \{z \in V|\ \phi(t,z) \in V \text{ for all } t \geq 0,\\ \phi(t,z) \to x_0 \text{ as } t \to \infty\},$$

the ***local unstable set***

$$W^u(x_0, V) = \{z \in V|\ \phi(t,z) \in V \text{ for all } t \leq 0,\\ \phi(t,z) \to x_0 \text{ as } t \to -\infty\},$$

the ***stable manifold***

$$W^s(x_0) = \{z|\ \phi(t,z) \to x_0 \text{ as } t \to \infty\},$$

the ***unstable manifold***

$$W^u(x_0) = \{z|\ \phi(t,z) \to x_0 \text{ as } t \to -\infty\}.$$

Suppose $x_0 = 0$, i.e. $f(0) = 0$, and $Df|_0$ has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_s$, $\mu_1, \mu_2, \dots, \mu_u$ where $\mathrm{Re}(\lambda_i) < 0$, $\mathrm{Re}(\mu_i) > 0$.

Assume $Df|_0$ is in Jordan form, with

$$Df|_0 = \begin{pmatrix} \mu\text{'s} & 0 \\ 0 & \lambda\text{'s} \end{pmatrix}$$

So if $f = Df|_0\, x$ (i.e. if f were linear), then the flow would look like this:
Let

$$E^u = \{(x_1, x_2, \dots, x_u, 0, \dots, 0)\}$$
$$E^s = \{(0, \dots, 0, x_{u+1}, \dots, x_n)\}$$

Solution to linear flow would be

$$x(t) = e^{(Df|_0 t} x_0.$$

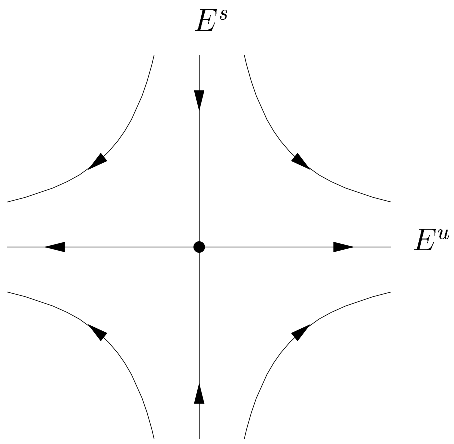

FIGURE 13. Stable and unstable manifolds of the linear system

For example, in $\mathbf{R}^2$,

$$Df|_0 = \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}.$$

So

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \exp\left(\begin{bmatrix} \mu & 0 \\ 0 & \lambda \end{bmatrix} t \right) \cdot \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} e^{\mu t} x_1(0) \\ e^{\lambda t} x_2(0) \end{pmatrix}.$$

So $W^s(0) = E^s$ and $W^u(0) = E^u$.

Given a (nonlinear) f as above, the $W^s(0)$, $W^u(0)$ are immersed $\mathcal{C}^k$-submanifolds, with $W^s(0)$ tangent to E^s at 0, $W^u(0)$ tangent to E^u at 0.

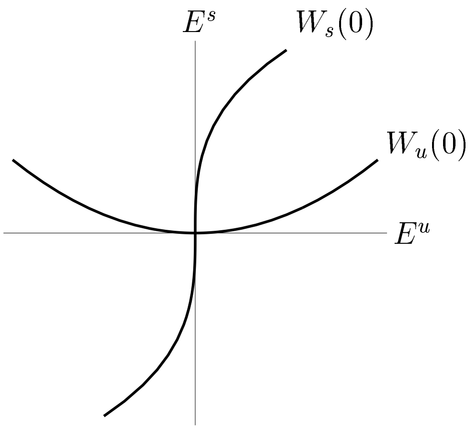

FIGURE 14. The stable and unstable manifolds are tangent to the linear subspaces.

The local version of the Stable/Unstable Manifold Theorem says that there exists a neighborhood V of x_0 such that $W^u(0, V)$ is the graph of a $\mathcal{C}^k$ function

$\zeta\colon E^u \cap V \to E^s$, i.e.

$$W^s(0,V) = \{\underbrace{\zeta(x_1, x_2, \dots, x_u, 0, \dots, 0)}_{\in E^s} + \underbrace{(x_1, x_2, \dots, x_u, 0, \dots, 0)}_{\in E^u}\}$$

where $\zeta(0) = 0$, $D\zeta|_0 = 0$.

Similarly, for $W^s(0,V)$, there exists a $\mathcal{C}^k$ function $\eta\colon E^s \cap V \to E^u$, where $W^s(0,V)$ is the graph of η, $\eta(0) = 0$, $D\eta|_0 = 0$.

Moreover, if f_α depends smoothly on α, then so do W^s, W^u (i.e. for α near α_0, f_α has a rest point near the rest point of f_{α_0} and stable/unstable manifolds depend smoothly on α), as long as the fixed point of f_α is hyperbolic.

Moreover, if $z \in V$, $z \notin W^s(0,V)$, then for some $t > 0$, $\phi(t,z) \notin V$, and if $z \notin W^u(0,V)$, for some some $t < 0$, $\phi(t,z) \notin V$.

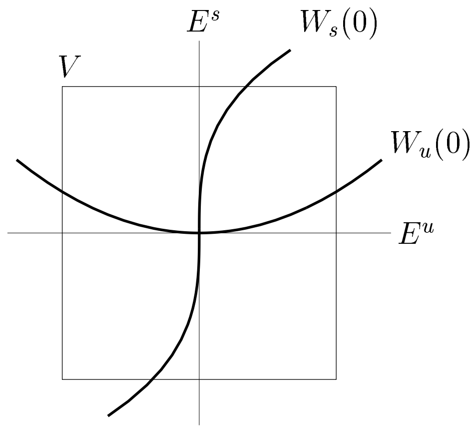

FIGURE 15. "The box"

Example: Look at $n = 2$, $u = s = 1$, f analytic. Then $W^s(0,V)$, $W^u(0,V)$ should be graphs of analytic functions.

$$Df|_0 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

with $\lambda > 0 > \mu$.

Linearization gives us that $\dot{x} = \lambda x$ and $\dot{y} = \mu y$, and we see that $E^u = \{(x,0)\}$ and $E^s = \{(0,y)\}$. Let

$$f(x,y) = (f_1(x,y), f_2(x,y)),$$

where

$$f_1(x,y) = \lambda x + \sum_{i+j\geq 2} a_{ij} x^i y^j,$$
$$f_2(x,y) = \mu y + \sum_{i+j\geq 2} b_{ij} x^i y^j$$

Look for $W^u(0,V)$ to be graph $\zeta\colon E^u \cap V \to E^s$, so

$$W^u(0,V) = \{(x,\zeta(x))|\, x(0) \in V\}$$

where $\zeta(x)$ is analytic, $\zeta(0) = 0$, $\left.\dfrac{d\zeta}{dx}\right|_0 = 0$.

So

$$\zeta(x) = \sum_{n=2}^{\infty} \alpha_n x^n = \alpha_2 x^2 + \alpha_3 x^3 + \cdots$$

If we have an initial condition in $W^u(0,V)$, the only way the solution leaves $W^u(0,V)$ is to leave V.

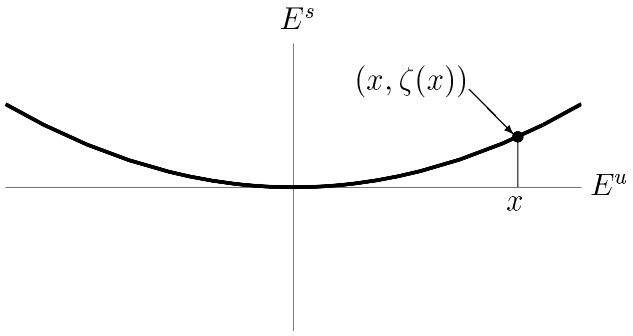

FIGURE 16. $\zeta(x)$ is a graph near the fixed point.

So vector field on $W^u(0,V)$ must be tangent to $W^u(0,V)$. At a point $(x,\zeta(x))$, the vector field is

$$f(x,\zeta(x)) = (f_1(x,\zeta(x)), f_2(x,\zeta(x))).$$

At $(x,\zeta(x))$, we want $f(x,\zeta(x))$ to be the tangent line to ζ. A vector in tangent line at $(x,\zeta(x))$ is $(1,\zeta'(x))$. So $(\zeta'(x),-1)$ is orthogonal to the tangent line, and we want that

$$f(x,\zeta(x)) \cdot (\zeta'(x),-1) = 0 \text{ for all } x$$

or

$$f_1(x,\zeta(x)) \cdot \zeta'(x) - f_2(x,\zeta(x)) = 0$$

Plug in:

$$\begin{aligned}&[\lambda x + a_{20}x^2 + a_{11}x(\alpha_2 x^2 + \alpha_3 x^2 + \cdots) + a_{02}(\alpha_2 x^2 + \alpha_3 x^3 + \cdots] \cdot \\ &\quad \cdot (2\alpha_2 x + 3\alpha_3 x^2 + \cdots) \\ &\quad - [\mu(\alpha_2 x^2 + \alpha_3 x^3 + \cdots) + b_{21}x^2 + b_{11}x(\alpha_2 x^2 + \alpha_3 x^3 + \cdots) + \cdots] = 0\end{aligned}$$

Some of the results will be:

$$\alpha_2 = \frac{b_{20}}{2\lambda - \mu}$$
$$\alpha_3 = \frac{-2a_{20}\alpha_2 + b_{11}\alpha_2 + b_{30}}{3\lambda - \mu}$$

9. More S/U Theorem (analytic version) (Oct. 16)

We can solve for those α's formally.

1. Do they converge?
2. Are solutions on graph of ζ?
3. Unique?

Try the $\mathcal{C}^k$ version, $\dot{x} = f(x)$, $x \in \mathbf{R}^2$, $f(0) = 0$.

Look at time T map associated to the solution flow ϕ for $\dot{x} = f(x)$. Let $F_T(x) = \phi(x, T)$. F_T^{-1} exists, since $F_T^{-1}(x) = \phi(x, -T)$. F_T is as differentiable as ϕ is in x. $F_T(0) = 0$, a rest point.

$$F_T^n(0) = \underbrace{F_T \circ F_T \circ \cdots \circ F_T}_{n \text{ times}} = \phi(x, nT)$$

by the group property.

Let

$$\mathcal{O}_{F_T}(x) = \{F_T^n(x) |\ n \in \mathbf{Z}\}.$$

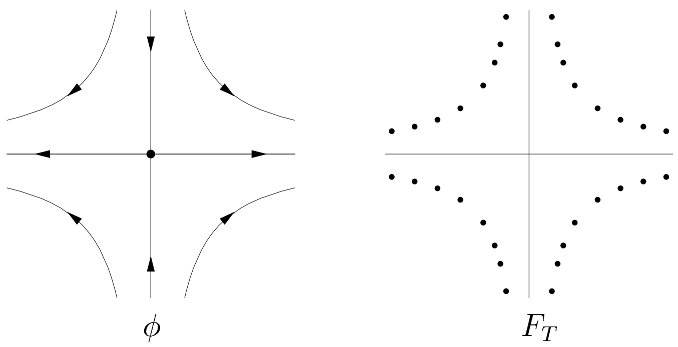

FIGURE 17. The flow ϕ and its associated map F_T.

Let

$$W^s_{F_T}(0) = \{x |\ \lim_{n\to\infty} F_T^n(x) = 0\},$$
$$W^u_{F_T}(0) = \{x |\ \lim_{n\to-\infty} F_T^n(x) = 0\},$$

and

$$W^s_{F_T}(0, V) = W^s_{F_T}(0) \cap \{x |\ F_T^n(x) \in V, n \geq 0\},$$

where V is a neighborhood of 0.

LEMMA 9.1. *Let $W^s_\phi(0)$ be a stable set of 0 for ϕ and $W^s_{F_T}(0)$ be a stable set of 0 for $F_T, T > 0$. Then*

$$W^s_\phi(0) = W^s_{F_T}(0)$$

Proof: Clearly, $W^s_\phi(0) \subseteq W^s_{F_T}(0)$, because if $x \in W^s_\phi(0)$ then $\phi(x,t) \to 0$ as $t \to \infty$ which means that $\phi(x, nT) \to 0$ as $n \to \infty$.

If $x \in W^s_{F_T}(0)$ then $F^n_T \to 0$ as $n \to \infty$ or $\phi(x, nT) \to 0$ as $n \to \infty$.

Note that for all $\epsilon > 0$, there is a $\delta > 0$ such that if $z \in \mathbf{R}^2$, $\|z\| < \delta$, then $\|\phi(z,t)\| < \epsilon$ for $0 < t \leq T$.

Fix $\epsilon > 0$, and choose N so large such that for all $n \geq N$, if $\|\phi(x, nT)\| < \delta$, then $\|\phi(x, nT + s)\| < \epsilon$, so that $\|\phi(x,t)\| < \epsilon$ for all $t > NT$. □

So we can deal with $W^s_\phi(0)$ and $W^s_{F_T}(0)$ similarly.

What is a hyperbolic fixed point for a map?

Suppose we have the same situation as above for $\dot{x} = f(x)$, with

$$Df|_0 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

How do we compute DF_T?

$$\dot{x} = f(x)$$
$$\dot{X} = Df|_x \, X$$

where X is an $n \times n$ matrix, with initial conditions $x(0) = x_0$ and $X(0) = I$. If $x(t), X(t)$ is a solution, then

$$X(t) = D_x\phi(x_0, t).$$

Recall, this comes from differentiating twice:

$$\begin{aligned} \frac{d}{dt} D_x\phi &= D_x(\frac{d}{dt})\phi \\ &= D_x f|_{\phi(x,t)} \, D_x\phi(x,t) \end{aligned}$$

Suppose we take $x(0) = 0$ and start at the rest point.

Solve $\dot{x} = f(x)$ part, have $x(t) = 0$. So

$$\dot{X} = Df|_{x(t)} \, X = Df|_0 \, X$$

and we get

$$\dot{X} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} X, \quad X(0) = I.$$

So

$$X(t) = \exp\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} t \right\},$$

$$D_x F_T|_0 = \exp\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} T \right\} = \begin{pmatrix} e^{\lambda T} & 0 \\ 0 & e^{\mu T} \end{pmatrix}.$$

The eigenvalues of $DF_T|_0$ are $e^{(\text{eigenvalues of } Df|_0)T}$.

So if λ is an eigenvalue of $Df|_0$ with real part > 0, then the corresponding eigenvalue for $DF_T|_0$ is $e^{\lambda T}$ with $\|e^{\lambda T}\| > 1$. If μ is an eigenvalue of $Df|_0$ with real part < 0, then the corresponding eigenvalue for $DF_T|_o$ 0 is $e^{\mu T}$ with $\|e^{\mu T}\| < 1$.

Definition 9.1. *Given $F\colon \mathbf{R}^n \to \mathbf{R}^n$, $F(x_0) = x_0$, F a diffeomorphism, then x_0 is **hyperbolic** iff none of the eigenvalues of $DF|_{x_0}$ has norm 1.*

Definition 9.2. *We call a fixed point x_0 of $F\colon \mathbf{R}^n \to \mathbf{R}^n$ a **sink** if there is a neighborhood U of x_0 such that for all $z \in U, F^n(z) \in U$, all $n \geq 0$, and*

$$\lim_{n\to\infty} F^n(z) = x_0.$$

Theorem 9.1. *If all the eigenvalues of $DF|_{x_0}$ are inside the unit circle, then x_0 is a sink.*

Theorem 9.2 (2-dim Stable/Unstable Manifold Theorem for Maps). *Suppose we have $F\colon \mathbf{R}^2 \to \mathbf{R}^2$ a diffeomorphism, $F(0) = 0$, and*

$$DF|_0 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \text{ with } |\lambda| > 1, |\mu| < 1.$$

Then there is a neighborhood V of 0 and $\zeta\colon [-\epsilon, \epsilon] \to \mathbf{R}$ with $W^u(0, V)$ the graph of ζ, i.e. $W^u(0, V) = \{(x, \zeta(x))\}$, and $\zeta(0) = 0$, $\zeta'(0) = 0$.

Steps:

1. Prove ζ is Lipschitz.
2. Bootstrap to get more smoothness.

Look at neighborhood of 0:

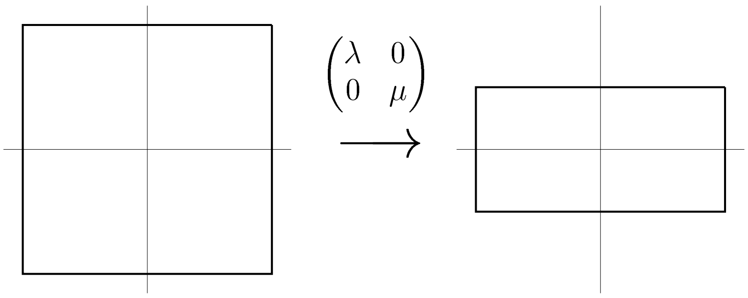

Figure 18. What the linear part of F does to the box

For small boxes, the map will look a lot like its linear part.

Note that $F(W^u(0)) = W^u(0)$.

Set up a map on graphs in the box and look for fixed points. (Graph Transform Method).

10. $\mathcal{C}^k$ version of S/U Theorem (Oct. 18)

Lemma 10.1. *Suppose F is a $\mathcal{C}^k$-diffeomorphism and $F(0) = 0$ is a hyperbolic fixed point. If $W^u(0, V)$ is the graph of a $\mathcal{C}^k$ function then $W^u(0)$ is a $\mathcal{C}^k$ immersed submanifold, i.e. local implies global.*

Proof: Recall our definition of a $\mathcal{C}^k$ immersed submanifold is that a neighborhood of each point in the manifold is (in nice coordinates) a $\mathcal{C}^k$ graph. So pick $z \in W^u(0)$, i.e. $F^n(z) \to 0$ as $n \to \infty$. So for any neighborhood V of 0, there is an N such that $F^{-n}(z) \in V$ for all $n \geq N$. So $F^{-N}(z) \in W^u(0, V)$, then $F^{-N}(z)$

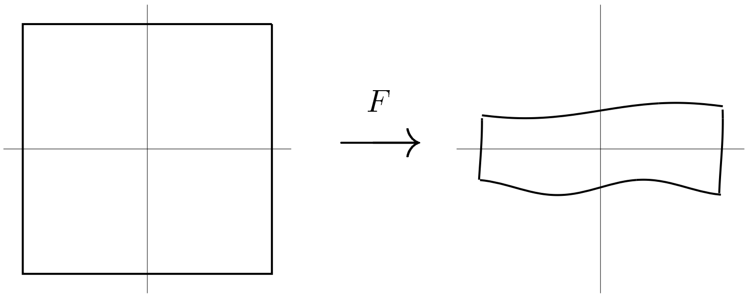

FIGURE 19. What F does to the box

is on graph of $\mathcal{C}^k$ function. Use F^{-N} to change coordinates, in coordinates in V, $W^u(0)$ near z is a graph. So $W^u(0)$ near z is a graph of a $\mathcal{C}^k$ function. $\square$

Let us set up the 2-dimensional version: $F\colon \mathbf{R}^2 \to \mathbf{R}^2$, $F(0) = 0$, with $F \in \mathcal{C}^2$.

$$DF|_0 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \lambda > 1 > \mu > 0$$

Our goal is to show that $W^u(0, V)$ is a graph of a function $\zeta\colon E^u \cap V \to E^s$. The idea is to set up a nice neighborhood V.

Iterate this process, and hope the graphs go to $W^u(0, V)$. The local unstable manifold has to map to itself. $W^u(0, V)$ should be the fixed point of this process.

Details.

1. Set up appropriate space of graphs on function (ζ). We want that F(graph in set) is a graph in set, and to preserve as much smoothness as possible.
2. Need our space to be complete.
3. Map on this space has a sink (is a contraction).
4. Need to show that $W^u(0, V)$ is a fixed point.

Start by choosing as neighborhood V of 0 that looks like $[-\epsilon_v, \epsilon_v] \times [-\epsilon_v, \epsilon_v]$, with ϵ_v chosen below.

Now look at $F(x, y) = (\lambda x, \mu y) + (g_1(x, y) + g_2(x, y))$. By the 2-dimensional Taylor's Theorem, we have $Dg_1|_0 = 0 = Dg_2|_0$, so Dg_1, Dg_2 are small near 0, so there is ϵ, K_1 such that if $\|(x, y)\| < \epsilon$, then

$$\|g_i(x, y)\| < K_1 \|(x, y)\|^2$$

Take $\epsilon_v < \epsilon_1$, and use that to get control of $F(V)$.

We want the right edge to map to the right edge, so we want:

$$\begin{aligned} &|\lambda\epsilon + g_1(\epsilon, y)| > \epsilon, \text{ for } -\epsilon < y < \epsilon \\ \text{i.e.,}\quad &|g_1(\epsilon, y)| < K_1 \sqrt{\epsilon^2 + y^2}^2 \text{ for } \epsilon < \epsilon_1 \\ &\qquad < K_1 \sqrt{2}\epsilon^2 \\ &\qquad < (\lambda - 1)\epsilon. \end{aligned}$$

So we need

$$\epsilon < \frac{\lambda - 1}{2K_1\sqrt{2}}, \text{ for } \lambda > 1.$$

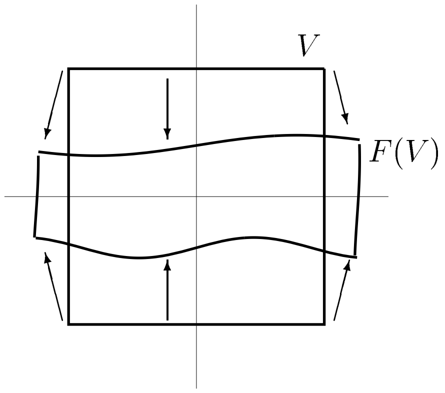

FIGURE 20. What the map F does to the box V

Take $\epsilon_v < \epsilon$.
For the top, we need

$$\begin{aligned} &|\mu\epsilon + g_2(x,\epsilon)| < \epsilon \text{ for } -\epsilon < x < \epsilon, \\ &\text{i.e. } \ |g_2(x,\epsilon)| < (1-\mu)\epsilon, \\ &\qquad k_1\sqrt{2}\epsilon^2 < (1-\mu)\epsilon, \\ &\qquad\qquad \epsilon < \frac{1-\mu}{2K_1\sqrt{2}} \quad (\mu < 1). \end{aligned}$$

So choose ϵ_v to be less than all of these.
Next, chose our set of graphs. Let

$$\begin{aligned} G_L = \{\zeta\colon E^u \cap V \to E^s |\ & x_1, x_2 \in [-\epsilon_v, \epsilon_v], \\ & |\zeta(x_1) - \zeta(x_2)| < L\,|x_1 - x_2|\} \end{aligned}$$

Assume that $L < 1$. Let

$$\mathcal{G}_L = \{A \subseteq V |\ A \text{ is the graph of } \zeta \in G_L\}$$

Define the graph transform: For $A \in \mathcal{G}_L$ let

$$\mathcal{F}(A) = \{F(x,y)|\ (x,y) \in A\} \cap V$$

We need that $\mathcal{F}(A) \in \mathcal{G}_L$. We will first show that $\mathcal{F}(A)$ is the graph of some function, and then that this function is in G_L.

Claim: for fixed $A \in \mathcal{G}_L$, for each $x \in [-\epsilon_v, \epsilon_v]$, there is a $y \in [-\epsilon_v, \epsilon_v]$ such that $(x,y) \in \mathcal{F}(A)$.

Graph of A is curve connecting left to right. So $\mathcal{F}(A)$ does the same thing for $F(V)$. So by conditions on $F(V)$, we know that for each $x \in [-\epsilon_v, \epsilon_v]$, there is a $y \in [-\epsilon_v, \epsilon_v]$ such that $(x,y) \in \mathcal{F}(A)$.

If the line containing x from top to bottom didn't intersect $\mathcal{F}(A)$, this would be a contradiction. But what if both (x,y_1) and $(x,y_2) \in \mathcal{F}(A)$?

Look at $F^{-1}(x,y_1), F^{-1}(x,y_2)$ in A. But we can write

$$F^{-1}(x,y) = (x/\lambda, y/\mu) + (\gamma_1(x,y), \gamma_2(x,y))$$

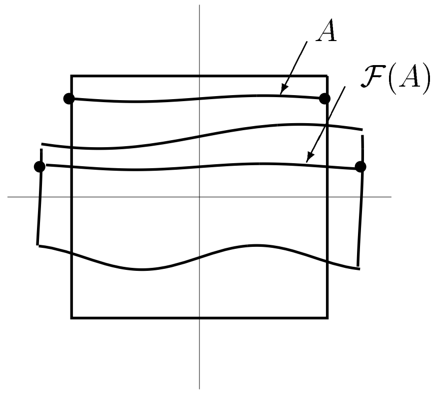

FIGURE 21. What F does to V and $\mathcal{F}$ does to A

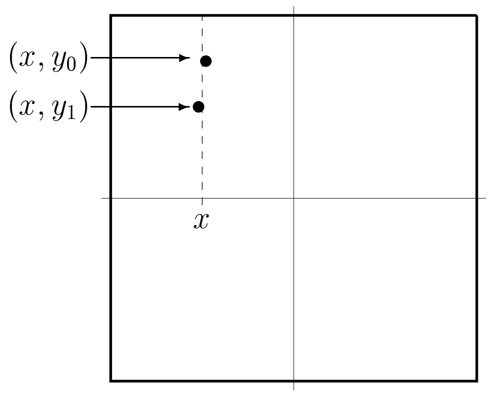

FIGURE 22. The points are "almost" on top of each other

where

$$DF^{-1}\big|_0 = (DF|_0)^{-1} = \begin{pmatrix} 1/\lambda & 0 \\ 0 & 1/\mu \end{pmatrix}$$

Then there is ϵ_2, K_2 such that if $\|(x,y)\| < \epsilon_2$, then

$$\|\gamma_1(x,y)\| < K_2 \|(x,y)\|^2 ,$$

replacing old ϵ_1 by ϵ_2, and K_1 by K_2 if necessary.

We expect that $F^{-1}(x, y_1)$ and $F^{-1}(x, y_2)$ are "almost" on top of each other, but this is bad because A is the graph of a Lipschitz function.

11. Continuation of the $\mathcal{C}^k$ proof (Oct. 21)

Now we know that each $\mathcal{F}(A)$ is a graph of a function, but we want that this function is in G_L.

At each $(x, y) \in V$, let

$$C_{(x,y)} = \{(x_1, y_1)| \ |y - y_1| < L|x - x_1|\}$$

Claim: If $(x, y) \in V$ and $F(x, y) \in V$ then

$$(F(C_{(x,y)} \cap V)) \cap V \subseteq C_{F(x,y)}.$$

We say then that F satisfies a cone condition. (This will be true provided that ϵ_v is sufficiently small.)

Proof: Suppose not, i.e.

$$|y - y_1| < L|x - x_1| \text{ for some } (x, y), (x_1, y_1) \in V,$$

but

$$|(\mu y + g_2(x, y)) - (\mu y_1 - g_2(x_1, y_1))| \geq L|(\lambda x + g_1(x, y)) - \lambda x_1 + g_1(x, y))|.$$

Now, $\mu|y - y_1| - |g_2(x, y) - g_2(x_1, y_1)| > L\lambda|x - x_1| - L|g_1(x, y) - g_1(x_1, y_1)|$

and

$$\begin{aligned} |g_2(x, y) - g_2(x_1, y_1)| &< \delta|x - x_1| + \delta|y - y_1| \\ |g_1(x, y) - g_1(x_1, y_1)| &< \delta|x - x_1| + \delta|y - y_1| \end{aligned}$$

where

$$\delta > \max\left\{\left|\frac{\partial g_i}{\partial x}\right|, \left|\frac{\partial f_i}{\partial x}\right|, i = 1, 2, \forall x, y \in V\right\}.$$

So

$$\begin{aligned} (\mu + \delta)|y - y_1| + \delta|x - x_1| &> L\lambda|x - x_1| - L\delta|x - x_1| - L\delta|y - y_1| \\ (\mu + \delta + L\delta)|y - y_1| &> (L\lambda - L\delta - \delta)|x - x_1| \end{aligned}$$

which gives us that

$$|y - y_1| > \frac{L\lambda - L\delta - \delta}{\mu + \delta + L\delta}|x - x_1|$$

Claim: By taking ϵ_v sufficiently small,

$$\frac{L\lambda - L\delta - \delta}{\mu + \delta + L\delta} > L$$

We know that $\dfrac{\partial g_i}{\partial x}, \dfrac{\partial g_i}{\partial y}$ are continuous (and 0) at 0. By taking ϵ_v sufficiently small, we can make δ as small as we like.

$$\frac{L\lambda - L\delta - \delta}{\mu + \delta + L\delta} \to L\frac{\lambda}{\mu} > L, \quad (\delta \to 0)$$

So if $A \in \mathcal{G}_L$, for each $(x, y) \in A$,

$$A \subseteq C_{(x,y)} \cap V \Rightarrow \mathcal{F}(A) \subseteq C_{F(x,y)} \cap V.$$

So $\mathcal{F}(A) \in \mathcal{G}_L$. $\square$

Claim: $\mathcal{F}$ is a contraction.

Given $\zeta_1, \zeta_2 \in G_L$, compute $\|\mathcal{F}(\zeta_1) - \mathcal{F}(\zeta_2)\|$. We want it to be less than

$$r \cdot \|\zeta_1 - \zeta_2\| = \sup_{x \in [-\epsilon_v, \epsilon_v]} |\zeta_1(x) - \zeta_2(x)| \quad (r < 1).$$

Fix $x \in [-\epsilon_v, \epsilon_v], (x, y_i) \in$ graph of ζ_i.

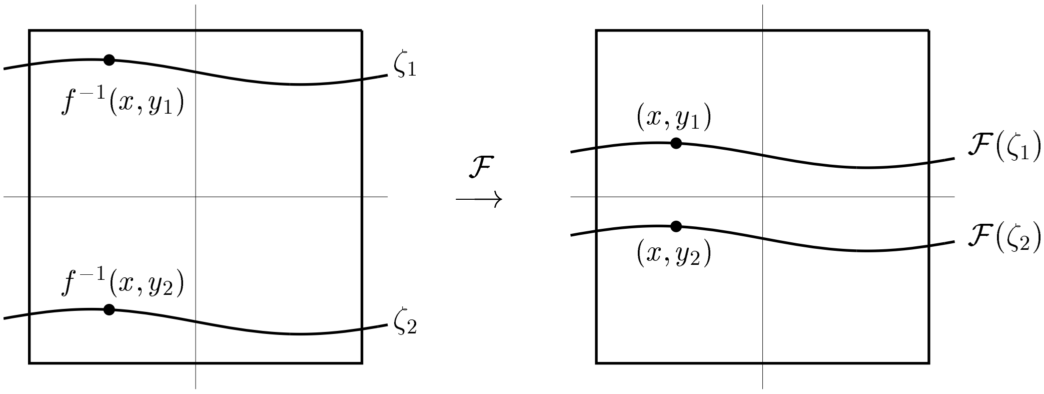

FIGURE 23. F is a contraction mapping.

$$F^{-1}(x,y_i) = (\frac{x}{\lambda} + \gamma_1(x,y_i), \frac{y_i}{\mu} + \gamma_2(x,y_i))$$

Estimate $|y_1 - y_2|$ compared to $\|\zeta_1 - \zeta_2\|$. First note that

$$\left|\left(\frac{x}{\lambda} + \gamma_1(x,y_1)\right) - \left(\frac{x}{\lambda} + \gamma_1(x,y_2)\right)\right| < \delta\,|y - y_1| \text{ where } \delta \geq \left\{\left|\frac{\partial \gamma_i}{\partial x}\right|, \left|\frac{\partial \gamma_i}{\partial y}\right|\right\}.$$

So we have

$$\left|\left(\frac{y_1}{\mu} + \gamma_2(x,y_1)\right) - \left(\frac{y_2}{\mu} + \gamma_2(x,y_2)\right)\right| \leq \|\zeta_1 - \zeta_2\| + L(\delta\,|y - y_1|)$$

which gives us

$$\begin{aligned}\frac{1}{\mu}|y_1 - y_2| - |\gamma_2(x,y_1) - \gamma_2(x,y_2)| &\leq \|\zeta_1 - \zeta_2\| + L\delta\,|y_1 - y_2| \\ \frac{1}{\mu}|y_1 - y_2| - \delta\,|y_1 - y_2| &\leq \|\zeta_1 - \zeta_2\| + L\delta\,|y_1 - y_2| \\ (\frac{1}{\mu} - \delta - L\delta)\,|y_1 - y_2| &\leq \|\zeta_1 - \zeta_2\| \\ |y_1 - y_2| &\leq \frac{1}{1/\mu - \delta - L\delta}\|\zeta_1 - \zeta_2\|\end{aligned}$$

Since the partials of γ_i are 0 at 0, take ϵ_v small so that

$$\frac{1}{1/\mu - \delta - L\delta} = r < 1 \text{ and } \mu < 1.$$

So $\mathcal{F}$ is a contraction. Now use the contraction mapping principle, and the fact that G_L is a metric space. So there must exist a fixed point for $\mathcal{F}$, ζ_u.

Claim: $\zeta_u = W^u(0,V)$.

Proof:

1. If $(x,y) \in$ graph ζ_u then $F^{-n}(x,y) \in V$ for all $n \geq 0$ and $F^{-n}(x,y) \to 0$ as $n \to \infty$.
2. If $(x,y) \notin$ graph ζ_u then there is an $n \geq 0$ such that $F^{-n}(x,y) \notin V$.

$\square$

Claim: For ϵ_v sufficiently small, there is an $r < 1$ such that if $(x, y) \in C_0$, then if $(x, y_1) = F^{-1}(x, y)$ then $|x_1| < r\,|x|$.

Proof: We know that $x_1 = x/\lambda + \gamma_1(x, y)$. Then

$$\begin{aligned} |\gamma_1(x,y)| &= |\gamma_1(x,y) - 0| \\ &\le \delta\,|x| + \delta\,|y| \\ &\le \delta\,|x| + \delta L\,|x|\,. \end{aligned}$$

So

$$|x_1| < \left|\frac{x}{\lambda}\right| + (\delta + \delta L)\,|x|$$

or

$$|x_1| < (\frac{1}{\lambda} + \delta + \delta L)\,|x|$$

$1/\lambda < 1$, so take ϵ_v small so that

$$\frac{1}{\lambda} + \delta + \delta L = r < 1.$$

We know that $\zeta_u(0) = 0$, because for and $\zeta \in G_L$, $\mathcal{F}^n(\zeta) \to \zeta_u$. If $\zeta(0) = 0$, then the value of $\mathcal{F}(\zeta)$ at 0 is also 0. And $\zeta_u \in G_L$, so the graph of ζ_u is in C_0 and if $(x, y) \in$ graph ζ_u, then $F^{-1}(x, y) \in$ graph ζ_u. So $F^{-n}(x, y) \in$ graph $\zeta_u \subseteq C_0$ for all $n \ge 0$.

So x-coordinate of $F^{-n}(x, y) \to 0$, but this is Lipschitz, so the y-coord $\to 0$ also. Thus the graph of $\zeta_u \subseteq W^u(0, V)$. $\square$

If you're not in the graph, the vertical distance increases, so eventually you leave V.

12. Smoothness (Oct. 28)

Start with $F\colon \mathbf{R}^2 \to \mathbf{R}^2$, a diffeomorphism, $F(0) = 0$, and

$$DF|_0 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \lambda > 1 > \mu > 0.$$

Lift to the unit tangent bundle

$$T^1\mathbf{R}^2 = \{((x, y), v)|\ v \in \mathbf{R}^2, \|v\| = 1\}$$

and

$$F_*((x,y), v) = \left(F(x,y), \frac{DF_{(x,y)}(v)}{\left\|DF_{(x,y)}(v)\right\|} \right).$$

At $x = y = 0$,

Eigendirections at $(x, y) = (0, 0)$ give the fixed points of F_*. Here $\theta = 0$ is stable in the θ-direction.

Compute $DF_*|_{((0,0),0)}$:

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ * & * & ? \end{pmatrix}$$

At $x = y = 0$, what does F_* do to $((0, 0), \theta)$?

Note: θ corresponds to $(\cos\theta, \sin\theta)$.

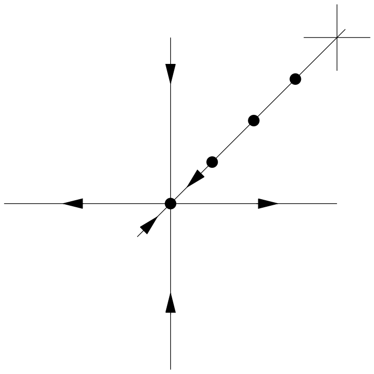

FIGURE 24. The fixed points for $(x, y) = (0, 0)$

θ-coordinate of image of $F_*((0,0),\theta)$ is

$$\frac{(\lambda\cos\theta, \mu\sin\theta)}{\sqrt{\lambda^2\cos^2\theta + \mu^2\sin^2\theta}}$$

and angle of image is

$$\arctan\left(\frac{\lambda\cos\theta}{\mu\sin\theta}\right),$$

i.e.

$$F_*((0,0),0) = \left((0,0), \arctan\left(\frac{\lambda\cos\theta}{\mu\sin\theta}\right)\right), \qquad \frac{\partial(\theta - \text{coord})}{\partial\theta} = \frac{\mu}{\lambda}.$$

Use the Lipschitz Unstable Manifold Theorem to get a 1-dimensional Lipschitz curve in $T^1\mathbf{R}^2$ through $((0,0),0)$ corresponding to unstable eigendirection.

We need to show that $W^u((0,0),0)$ is the derivative of $W^u(0,0)$, i.e. for $z \in W^u(0,0)$, the tangent (unit) vector of $W^u(0,0)$ at z is the corresponding point $(z,\theta) \in W^u((0,0),0)$.

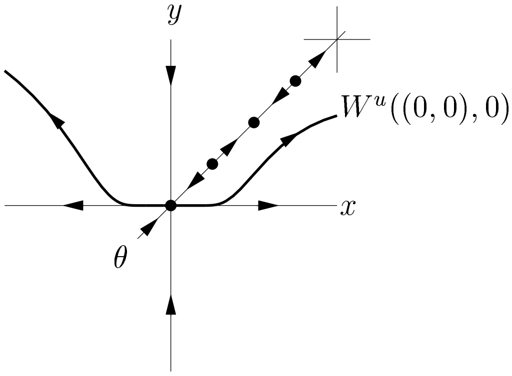

FIGURE 25. The unstable manifold at $((0,0),0)$. The picture may be a bit unclear. The θ-axis moves back into the 3rd dimension. Recall that there are fixed points on the θ-axis, alternating between sink and source.

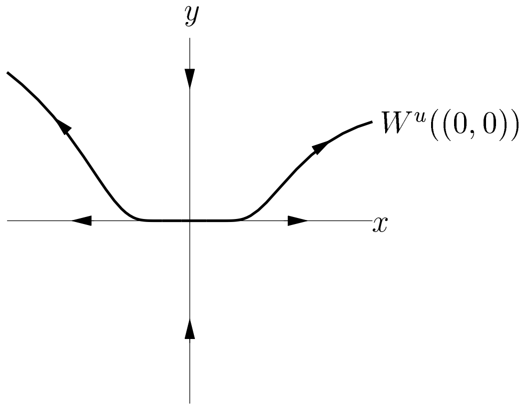

FIGURE 26. The unstable manifold at $(0,0)$ (what we get when we restrict to the plane

For $(x,\zeta(x)) \in W^u(0,V)$, define

$$C(x) = \left\{ v \,|\; v = \frac{\left(1, \lim_{n\to\infty} \dfrac{\zeta(x_n)-\zeta(x)}{x_n - x}\right)}{\left\| \left(1, \lim_{n\to\infty} \dfrac{\zeta(x_n)-\zeta(x)}{x_n - x}\right) \right\|} \right\}.$$

where $x_n \to x$. $W^u(0,V)$ is Lipschitz.

Claim: if $F(x,\zeta(x)) = (x_1, \zeta(x_1))$, then

$$F_*(C(x)) = C(x_1)$$

and

$$DF|_{(x,\zeta(x))} \left(1, \lim_{n\to\infty} \frac{\zeta(x_n)-\zeta(x)}{x_n - x}\right) = \lim_{n\to\infty} DF|_{(x,\zeta(x))} \left(1, \frac{\zeta(x_n)-\zeta(x)}{x_n - x}\right). \tag{31}$$

Proof: If we let

$$DF|_{x,\zeta(x))} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we get in equation (31):

$$\lim_{n\to\infty} \left(a + b\frac{(\zeta(x_n)-\zeta(x))}{x_n - x}, c + d\frac{(\zeta(x_n)-\zeta(x))}{x_n - x} \right) \underset{\text{(up to length)}}{=} \left(1, \frac{c(x_n - x) + d(\zeta(x_n)-\zeta(x))}{a(x_n - x) + b(\zeta(x_n)-\zeta(x))}\right).$$

We know that $x_n \to x$, so $F(x_n, \zeta(x_n)) \to (x_1, \zeta(x_1))$.

Look at $\{F(x_n, \zeta(x_n))\}$:

Use X, Y as projection onto x, y coordinates. Check:

$$\frac{Y(F(x_n,\zeta(x_n))) - Y(F(x,\zeta(x)))}{X(F(x_n,\zeta(x_n))) - X(F(x,\zeta(x)))}$$

$$= \frac{Y(F(x,\zeta(x))) + Y(DF|_{(x,\zeta(x))}\,(x_n - x, \zeta(x_n)-\zeta(x))) + YR - Y(F(x,\zeta(x)))}{X(F(x,\zeta(x))) + X(DF|_{(x,\zeta(x))}\,(x_n - x, \zeta(x_n)-\zeta(x))) + XR - X(F(x,\zeta(x)))}.$$

R is the higher order terms, so we can forget about it when we take limit, so we get

$$\lim_{n\to\infty}\left(\frac{c(x_n-x)+d(\zeta(x_n)-\zeta(x))}{a(x_n-x)+b(\zeta(x_n)-\zeta(x))}\right),$$

thus

$$F_*\colon C(x)\to C(x_1).$$

□

What does this say?

So if F is $\mathcal{C}^3$, then we have that $W^u(0)$ is $\mathcal{C}^{1,\mathrm{Lip}}$. If F is $\mathcal{C}^\infty$, then $W^u(0)$ is $\mathcal{C}^\infty$. In fact, we could show that $W^u(0)$ is as smooth as F, i.e. if F is $\mathcal{C}^k$, then $W^u(0)$ is $\mathcal{C}^k$.

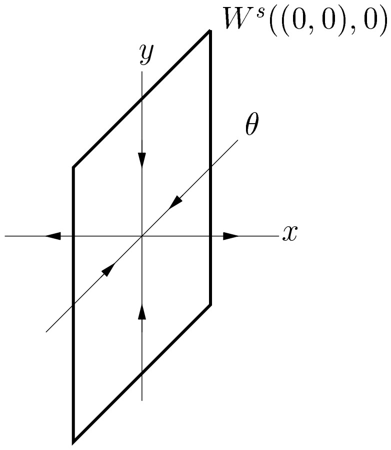

FIGURE 27. The stable manifold of $((0,0),0)$.

What is the geometric interpretation?

Remarks:

1. Stable manifold works the same. Replace F by F^{-1}.
2. Higher dimensional, we showed $\mathcal{F}(\text{graph})$ is a graph. In higher dimensions, if A is Lipschitz graph $E_u\to E_s$, we need

$$F(A)\cap E_s+z\neq 0\Leftrightarrow A\cap F^{-1}(E_s+z)\neq 0.$$

Near the fixed point, $F^{-1}(E_s+z)$ is a perturbation of E_s+w for some w. So use transversality of intersection of E_s+w with A.

13. The Stable/Unstable Manifold MetaTheorem (Oct. 30)

THEOREM 13.1. *Given smooth ($\mathcal{C}^k$,$\mathcal{C}^\infty$) vector field $F\colon\mathbf{R}^n\to\mathbf{R}^n$, $F(0)=0$. $\lambda_1,\lambda_2,\dots,\lambda_n$ are the eigenvalues of $DF|_0$ (with redundancy), and $E_1,E_2,\dots,E_n$*

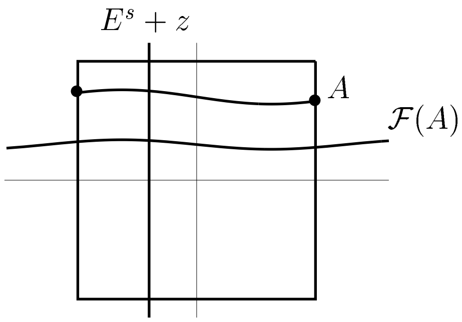

FIGURE 28. Perturbation of the stable manifold

are the corresponding normalized e-spaces, e.g. if

$$DF|_0 = \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & 1 & & & \\ & 0 & \lambda_2 & & & \\ & & & a & b & \\ & & & -b & a & \\ & & & & & \ddots \end{pmatrix},$$

then

$$\begin{aligned} E_1 &= \{(x_1, 0, \ldots, 0)\}, \\ E_2 &= \{(0, x_2, x_3, \ldots, 0)\}, \ldots \end{aligned}$$

The E_i are invariant subspaces for $DF|_0$.

For $r \le 0$, there is an invariant manifold $W_r^s(0)$ tangent to $E_1 \oplus \cdots \oplus E_m$, where Re λ_1,Re $\lambda_2, \ldots,$ Re $\lambda_m \le r$.

First, $\dim W_r^s(0) = \dim(E_1 \oplus \cdots \oplus E_m)$*. Also, if $r < 0, r <$ Re $\lambda_i < 0$ for some i, then $W_r^s(0)$ is called a* ***strong stable manifold*** *and denoted $W_r^{ss}(0)$. If $r = 0$ and there is some eigenvalue, λ_i, with Re $\lambda_i = 0$, then $W_r^s(0)$ is called the* ***center stable manifold*** *and denoted $W_r^{cs}(0)$.*

Same for $r \ge 0$, we get invariant manifolds associated with eigenvalues λ_j with Re $\lambda_j \ge r$. If $r > 0$, and there is λ_j with $0 <$ Re $\lambda_j < r$, then $W_r^u(0)$ is called the ***strong unstable manifold****, and denoted $W_r^{su}(0)$. If $r = 0$ and there is λ_j with Re $\lambda_j = 0$, $W_r^u(0)$ is called the* ***center stable manifold****, and denoted $W_r^{cu}(0)$.*

A ***center manifold*** *is defined as*

$$W_r^{sc}(0) \cap W_r^{uc}(0),$$

and is denoted $W^c(0)$. (It is tangent to e-spaces for eigenvalues with real part 0.)

Remark: Proofs use same sort of ideas (set contraction).

Example:

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = -y \end{cases},$$

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 \\ -y \end{pmatrix},$$

$$DF|_0 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

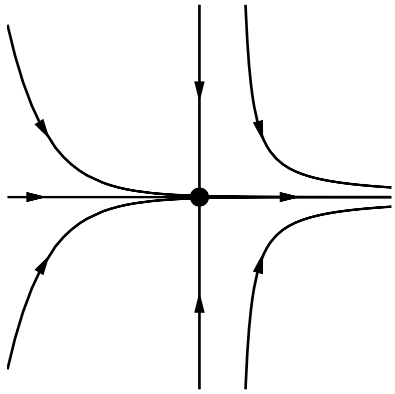

FIGURE 29. A center manifold

$W^{ss}(0)$ is the y-axis, $W^{cs}(0)$ is all of $\mathbf{R}^2$, so we have a lot of choices for $W^c(0)$. Choose any orbit on the left, and the x-axis on the right. So center manifolds are not unique, as opposed to the hyperbolic case.

Remarks:

1. Warning: These center manifolds are not necessarily as smooth as F.
2. The hyperbolic case is generic: Given a vector field with fixed point at 0 you can "expect" it to be hyperbolic, i.e. in the set of vector fields with fixed point at 0, the vector fields with hyperbolic fixed points at 0 form an open dense set in the $\mathcal{C}^1$ topology.

   Proof: If F has hyperbolic fixed point at 0 and G is sufficiently close to F and G has a fixed point at 0 (close in $\mathcal{C}^1$, so $DF|_0 \approx DG|_0$, so eigenvalues are close), so if none of the eigenvalues of $DF|_0$ are on the imaginary axis, and G is close enough, the same holds true for G. If 0 is not hyperbolic for F then $DF|_0$ has a Jordan block with 0 on the diagonal. Simply add ϵI_n to F, then

   $$D(F + \epsilon I_n)|_0 = DF|_0 + \epsilon I_n,$$

   so the diagonal elements will be nonzero if ϵ is sufficiently small. □

So why do we need center manifolds at all? If we're dealing with a one-parameter family of vector fields F_α (all of whom have a fixed point at 0, if when $\alpha = 0, DF_\alpha|_0$ is hyperbolic with s-dimensional stable and $(n-s)$-dimensional unstable, but $DF_1|_0$ has $(s+1)$-dimensional stable and $(n-s-1)$-dimensional unstable, then for some $\alpha \in [0,1]$, $DF_\alpha|_0$ is not hyperbolic. We'll talk a lot more about this when we do bifurcation theory.

CHAPTER 3

Using maps to understand flows

1. Periodic orbits and Poincaré sections (Nov. 1)

The next most complicated invariant set (after the fixed point) is the periodic orbit. For example, if ϕ is a flow on $\mathbf{R}^n$, then $x_0 \in \mathbf{R}^n$ is called a **periodic point with least period** T if

$$\phi(x_0, T) = x_0$$
$$\phi(x_0, t) \neq x_0 \text{ for } 0 < t < T.$$

Of course, we see that $\phi(x_0, nT) = x_0$ for all $n \in \mathbf{N}$. We call $\mathcal{O}(x_0)$ a **periodic orbit**. $\mathcal{O}(x_0)$ is the smooth image of a circle in $\mathbf{R}^n$, so the dynamics of $\phi|_{\mathcal{O}(x_0)}$ are very easy.

What about near periodic orbits?

Idea. (due to Poincaré)

Let Σ be a $(n-1)$-dimensional surface (copy of $(n-1)$-dimensional disk) that contains x_0 such that for all $x \in \Sigma$, $F(x)$ is not tangent to Σ.

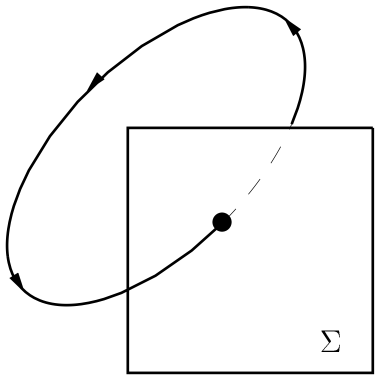

FIGURE 1. A Poincaré section

Σ is called a **Poincaré section** for ϕ. Define a map $P\colon \Sigma \to \Sigma$. For each $x \in \Sigma$, let

$$\tau(x) = \{t |\ t > 0, \phi(x, t) \in \Sigma\}.$$

If $\phi(x, t) \notin \Sigma$, let $\tau(x) = \infty$. By continuity with respect to initial conditions, if x is near enough to x_0, then $\tau(x) < \infty$.

We need to show that $\tau(x)$ is a smooth function in a neighborhood of x_0, ($\tau\colon \Sigma \to \mathbf{R}$).

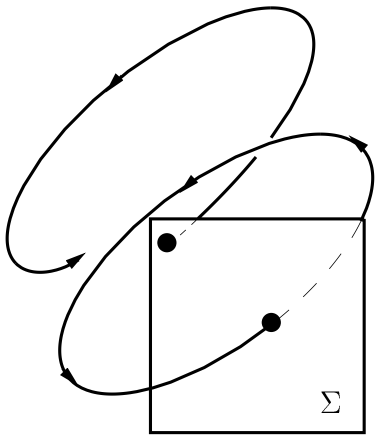

FIGURE 2. If the initial condition is far enough away, it may not actually return to Σ.

We know that $\tau(x_0) = T$, so make Σ smaller if necessary so that $\tau(x) \in [T-1, T+1]$. (Local theory near periodic orbits)

Next choose coordinates so that $x_0 = 0, \Sigma \subseteq \{(0, x_2, \ldots, x_n)\}$ in a neighborhood of x_0.

Let $f\colon \mathbf{R}^{n-1} \times \mathbf{R} \to \mathbf{R}$,

$$f(x_2, \ldots, x_n, t) = \pi_1 \circ \phi(0, x_2, \ldots, x_n, t)$$

where π_1 is restriction to the 1st coordinate. This is defined for $x_2, \ldots, x_n$ near 0 and t near T.

Look at the level set $f^{-1}(0)$. The Implicit Function Theorem implies that there is a $\tau(x_2, \ldots, x_n)$ such that

$$f(x_2, \ldots, x_n, \tau(x_2, \ldots, x_n)) = 0$$

provided that

$$\left.\frac{\partial f}{\partial t}\right|_{(0,T)} \neq 0.$$

This will work because

$$\left.\frac{\partial f}{\partial t}\right|_{(0,T)} = \frac{\partial}{\partial t}\pi_1(\phi(0,T)) = \pi_1 \circ F(\phi(0,T))$$

and $F(x_0)$ is not tangent to Σ.

So $\tau(x)$ is smooth on Σ near x_0.

Define $P\colon \Sigma \to \Sigma$,

$$P(x_2, \ldots, x_n) = \pi_\Sigma \circ \phi(0, x_2, \ldots, x_n, \tau(x_2, \ldots, x_n))$$

which is defined and smooth on a neighborhood of x_0 in Σ.

P is the **first return map** or **Poincaré Map**.

Remarks:

1. $P(x_0) = x_0$, so P has a fixed point at $\mathcal{O}(x_0) \cap \Sigma$.
2. P^{-1} exists (follow flow backwards). Of course P^{-1} is smooth, so P is a diffeomorphism on some neighborhood of x_0 in Σ.

Let us denote

$$W^s(\mathcal{O}(x_0)) = \{z \in \mathbf{R}^n | \max_{y \in \mathcal{O}(x_0)} \|\phi(z,t) - y\| \to 0, t \to \infty\},$$
$$W^u(\mathcal{O}(x_0)) = \{z \in \mathbf{R}^n | \max_{y \in \mathcal{O}(x_0)} \|\phi(z,t) - y\| \to 0, t \to -\infty\}.$$

Remark: This does not require that $\phi(z,t) \approx \phi(y,t)$ for some $y \in \mathcal{O}(x_0)$ when t is large, just that $\phi(z,t)$ is close to the set $\mathcal{O}(x_0)$ (i.e. not necessarily "in phase".)

DEFINITION 1.1. *We say that $\mathcal{O}(x_0)$ is a **sink** if $W^s(\mathcal{O}(x_0))$ contains an open neighborhood of $\mathcal{O}(x_0)$, a **source** if $W^u(\mathcal{O}(x_0))$ contains an open neighborhood of $\mathcal{O}(x_0)$.*

THEOREM 1.1. *Let x_0, ϕ, Σ, P be as above and $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$ be the eigenvalues of $DP|_{x_0}$ (perhaps with redundancy).*

1. *If $\|\lambda_i\| < 1$ for $i = 1, \dots, n$, then $\mathcal{O}(x_0)$ is a sink.*
2. *If $\|\lambda_i\| > 1$ for $i = 1, \dots, n$, then $\mathcal{O}(x_0)$ is a source.*
3. *If $\|\lambda_1\|, \dots, \|\lambda_s\| < 1$, and $\|\lambda_{s+1}\|, \dots, \|\lambda_{n-1}\| > 1$, i.e. x_0 is a hyperbolic fixed point for P and if E^s is the direct sum of the generalized e-spaces for $\lambda_1, \lambda_2, \dots, \lambda_s$ and E^u is the direct sum of the generalized e-spaces for $\lambda_{s+1}, \dots, \lambda_{n-1}$, then $W^s(\mathcal{O}(x_0))$ and $W^u(\mathcal{O}(x_0))$ are immersed submanifolds as smooth as ϕ of dimension $E^s + 1$ and $E^u + 1$ respectively, and $W^s(\mathcal{O}(x_0)) \cap \Sigma$ is tangent to E^s at x_0, and $W^u(\mathcal{O}(x_0)) \cap \Sigma$ is tangent to E^u at x_0.*

Proof:

1. Look at picture.

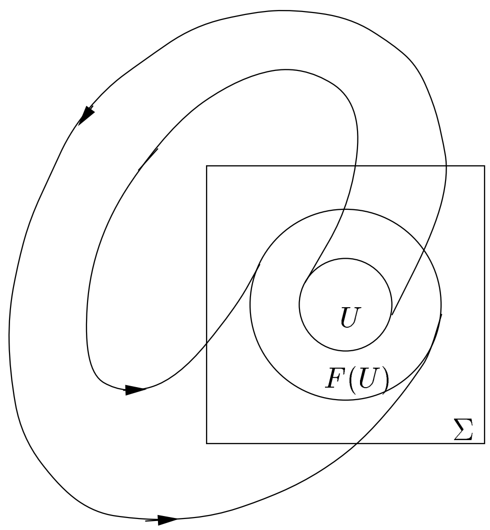

FIGURE 3. Poincaré map of a neighborhood

2. Same picture in backwards time.

FIGURE 4. Just as in the flow, the stable manifold shrinks in the Poincaré map and the unstable manifold grows

3. We have by transversality,

$$\dim W^s(\mathcal{O}(x_0)) + \dim W^u(\mathcal{O}(x_0)) = n + 1.$$

$\square$

Note: The periodic orbit is a subset of the intersection

$$\mathcal{O}(x_0) \subseteq W^u(\mathcal{O}(x_0)) \cap W^s(\mathcal{O}(x_0))$$

They intersect transversally at orbit, but the following picture is possible:

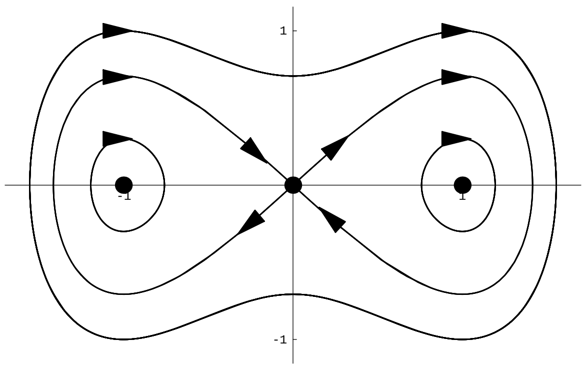

FIGURE 5. The fixed point at the origin has two homoclinic orbits in this picture.

Two questions:

1. What of this theory is independent of the choice of Σ?
2. Can you ever compute the eigenvalues for $DP|_{x_0}$, which is the eigenvalues of $D_x\phi(x_0, T)$?

2. Computing Floquet Multipliers (Nov. 4)

As before, we have vector field F, flow ϕ, periodic orb IR $\mathcal{O}(x_0)$, Poincaré section Σ, and Poincaré map $P\colon \Sigma \to \Sigma$.

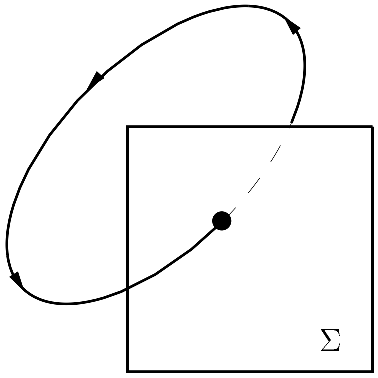

FIGURE 6. Poincaré section

The behavior of $\mathcal{O}(x_0)$ is determined by the eigenvalues of $DP|_{x_0}$.

The first problem is the independence of the choice of Σ. We could choose two Σ's, $P_1\colon \Sigma_1 \to \Sigma_1$, $P_2\colon \Sigma_2 \to \Sigma_2$.

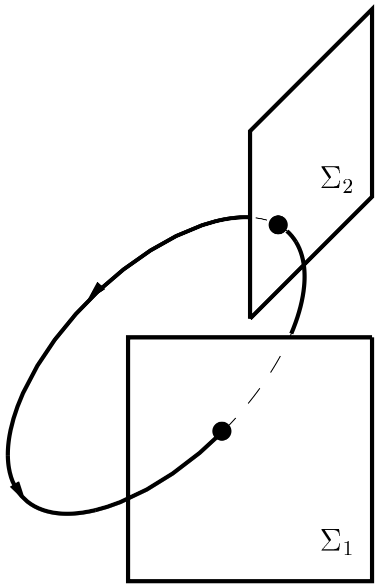

FIGURE 7. Is the Poincaré map independent of the choice of section?

Are Floquet multipliers independent of choice of Σ?

Idea. Define

$$\begin{aligned} p_1 : \Sigma_1 &\to \Sigma_2 \\ x &\mapsto \phi(x, \tau_1(x)) \end{aligned}$$

where

$$\tau_1(x) = \min\{t \geq 0 |\ \phi(x, \tau(x)) \in \Sigma_2\}.$$

As before, p_1 is defined near x_0 and as smooth as Σ and ϕ. Also, is invertible:

$$p_1^{-1}(x) = \phi(x, \tau_1^{-1}(x)) \text{ where } x \in \Sigma_2.$$

and

$$\tau_1^{-1}(x) = \max\{t < 0 |\ \phi(x, t) \in \Sigma_1\}.$$

Then our claim is that $P_1 = p_1^{-1} \circ P_2 \circ p_1$. This is the group property of ϕ. So

$$P_1|_{x_0} = Dp_1^{-1}\big|_{x_1} \cdot DP_2|_{x_1} \cdot Dp_1|_{x_0} .$$

But

$$D(p_1^{-1})\big|_{x_1} = (Dp_1)|_{x_0}{}^{-1}.$$

So $DP_1|_{x_0}$ is similar to $DP_2|_{x_1}$, and in particular these two matrices have the same eigenvalues.

The same idea works if $x_0 = x_1$:

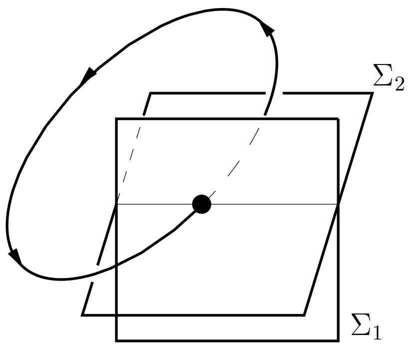

FIGURE 8. We assume that the Poincaré sections intersect at x_0.

Define

$$\begin{aligned} p_1 : \Sigma_1 &\to \Sigma_2 \\ x &\mapsto \phi(x, \tau_1(x)) \end{aligned}$$

where $\tau_1(x) = t$ near 0 such that $\phi(x, t) \in \Sigma_2$.

So the eigenvalues of the return map are invariants of the orbit. But we still need a way to computer the eigenvalues of the Poincaré maps.

Claim: The eigenvalues of $D_x\phi|_{(x_0,T)}$ are $1, \lambda_1, \ldots, \lambda_{n-1}$ where $\lambda_1, \ldots, \lambda_{n-1}$ are the Floquet multipliers.

Proof: The time T map takes a neighborhood of x_0 to a neighborhood of x_0. But $\phi(\cdot, T)$ preserves the periodic orbit $\mathcal{O}(x_0)$.

Let $\gamma(t) = \phi(x_0, t)$ so $\gamma \colon \mathbf{R} \to \mathbf{R}^n$.

Then

$$\begin{aligned}\phi(\gamma(t),T) &= \phi(\phi(x_o,t),T)\\ &= \phi(x_0,t+T)\\ &= \phi(x_0,t)\\ &= \gamma(t),\end{aligned}$$

i.e. all points on γ are fixed under $\phi(\cdot,T)$.

So

$$\frac{d}{dt}\phi(\gamma(t),T) = D_x\phi|_{(\gamma(t),T)} \cdot \left.\frac{d\gamma}{dt}\right|_{\tau}$$

Also,

$$\frac{d}{dt}\phi(\gamma(t),T) = \left.\frac{d}{dt}\gamma\right|_{t}.$$

If we let $t=0$, $(\gamma(0)=x_0)$

$$D_x\phi|_{(x_0,T)} \cdot \left.\frac{d\gamma}{dt}\right|_0 = \left.\frac{d\gamma}{dt}\right|_0,$$

i.e. $\left.\frac{d\gamma}{dt}\right|_0$ is an e-vector with eigenvalue 1.

Also,

$$\left.\frac{d\gamma}{dt}\right|_{t=0} = F(\gamma(0)) = F(x_0),$$

so

$$D_x\phi|_{(x_0,T)}\ F(x_0) = F(x_0).$$

We need that the rest of the eigenvalues of $D_x\phi|_{(x_0,T)}$ are eigenvalues of $DP|_{x_0}$ for $P\colon\Sigma\to\Sigma$. (i.e. Poincaré section)

Choose nice variables so that $F(x_0)=(1,0,\ldots,0)$, and then

$$D_x\phi|_{(x_0,T)} = \begin{pmatrix} 1 & ? & \cdots & ? \\ 0 & & \uparrow & \\ \vdots & \leftarrow & B & \rightarrow \\ 0 & & \downarrow & \end{pmatrix}.$$

From linear algebra, we know that the eigenvalues of B are the rest of the eigenvalues of $D_x\phi$, so

$$D_x\phi|_{(x_0,T)}\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = \begin{pmatrix} x_1 + ?x_2 + ?x_3 + \cdots + ?x_n \\ B\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\end{pmatrix} = \lambda\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}.$$

Next, choose $\Sigma = \{(0,x_2,\ldots,x_n)\}$ where in these nice coordinates, $x_0=0$. Let $\tau\colon\Sigma\to\mathbf{R}$ be the first return time map, so Poincaré map $P\colon\Sigma\to\Sigma$ is

$$P(x_1,x_2,\ldots,x_n) = \pi_1^*\circ\phi((0,x_2,\ldots,x_n),\tau(0,x_2,\ldots,x_n)),$$

where

$$\pi_1^*(x_1,x_2,\ldots,x_n) = (x_2,\ldots,x_n).$$

Extend τ to neighborhood of x_0 in $\mathbf{R}^n$ continuously so that $\tau(x_1, x_2, \ldots, x_n)$ is the time near T such that

$$\phi(x_1, x_2, \ldots, x_n, \tau(x_1, x_2, \ldots, x_n)) \in \Sigma.$$

Compute the $(n-1)$-dimensional vector

$$\frac{\partial P}{\partial x_2} = \pi_1^*(\frac{\partial}{\partial x_2}(\phi(0, x_2, \ldots, x_n), \tau(0, x_2, \ldots, x_n))).$$

$$\begin{aligned}
&\frac{\partial}{\partial x_2}\phi(0, x_2, \ldots, x_n, \tau(0, x_2, \ldots, x_n)) \\
&\quad = \left.\frac{\partial \phi}{\partial x_2}\right|_{(0,x_2,\ldots,x_n,\tau(0,x_2,\ldots,x_n))} + \left.\frac{\partial \phi}{\partial t}\right|_{(0,x_2,\ldots,x_n,\tau(0,x_2,\ldots,x_n))} \cdot \left.\frac{d\tau}{dx_2}\right|_{(0,x_2,\ldots,x_n)}.
\end{aligned}$$

Evaluate this at $x_0 = 0$ so that $\tau(0) = T$.

$$\begin{array}{ccccc}
\left.\dfrac{\partial \phi}{\partial x_2}\right|_{(x_0,T)} & + & \left.\dfrac{d\phi}{dt}\right|_{(x_0,T)} & \cdot & \left.\dfrac{d\tau}{dx_2}\right|_{x_0} \\
n \text{ vector} & & n \text{ vector} & & \text{scalar} \\
\left.\dfrac{\partial \phi}{\partial x_2}\right|_{(x_0,T)} & + & \underset{\text{vector field at } x_0}{(1, 0, \ldots, 0)} & \cdot & \left.\dfrac{d\tau}{dx_2}\right|_{x_0}.
\end{array}$$

So the last $(n-1)$ components of $\left.\dfrac{\partial}{\partial x_2}\phi(\cdot, \tau(0))\right|_{x_0} = \left.\dfrac{\partial P}{\partial x_2}\right|_{x_0}$ are the same as the last $n-1$ components of $\left.\dfrac{\partial \phi}{\partial x_2}\right|_{(x_0,T)}$.

The proof works exactly the same for $x_3, \ldots, x_n$, so $DP|_{x_0} = B$ in these nice coordinates. Thus the eigenvalues of P at x_0 are the same as B. ☐

This shifts the problem. Now how do we compute the eigenvalues of $D_x\phi|_{(x_0,T)}$?

There's only one way to get at $D_x\phi$ (unless we have an explicit form for ϕ). Recall the variational equation:

$$\begin{aligned}
\dot{x} &= F(x), & x(0) &= x_0 \\
\dot{X} &= DF|_{x(t)}\, X, & X(0) &= I
\end{aligned}$$

where

$$\begin{aligned}
x(t) &= \phi(x_0, t) \quad \text{is the periodic orbit, and} \\
X(t) &= D_x\phi|_{\phi(x_0,t)}
\end{aligned}$$

Verify that $\left.\dfrac{\partial \phi}{\partial t}\right|_{(x_0,t)} = F(\phi(x_0, t))$ by differentiating with respect to x. So, can we solve this in general? There are some situations where we can:

1. Find an explicit expression for the periodic orbit $\phi(x_0, t)$. (Easier, in general, than finding formula for $\phi(x, t)$).
2. $\dot{X} = DF|_{\phi(x_0,t)}\, X$ has the form $\dot{X} = A(t)X$ where $A(t)$ is an $n \times n$ matrix, "nice" in a sense to be made precise later.

3. More computation of Floquet multipliers (Nov. 6)

So when can we compute these Floquet multipliers? Let $A(t) = DF|_{\phi(x_0,t)}$, so that we have the equations

$$\begin{aligned} \dot{x} &= F(x), \qquad & x(0) &= x_0 \\ \dot{X} &= A(t)X, \qquad & X(0) &= I. \end{aligned}$$

There are some cases where we can solve this system:

1. $A(t) = A_0$, a constant matrix. In this case the solution is
$$X(t) = e^{A_0 t}.$$
2. If $\dim n = 1$, A is a 1×1 matrix, and $\dot{X} = a(t)X$. Then
$$X(t) = e^{\int_0^t a(s)\,ds}.$$
3. If
$$A(t) = \begin{pmatrix} a_1(t) & & & \\ & a_2(t) & & \\ & & a_3(t) & \\ & & & \ddots \end{pmatrix},$$
i.e. A is diagonal, then
$$X(t) = \begin{pmatrix} \exp\left\{\int_0^t a_1(s)\,ds\right\} & & & \\ & \exp\left\{\int_0^t a_2(s)\,ds\right\} & & \\ & & \exp\left\{\int_0^t a_3(s)\,ds\right\} & \\ & & & \ddots \end{pmatrix}$$

We would want that this solution would always work (even in the non-diagonal case). Since

$$\dot{X} = \lim_{\Delta t \to 0} \frac{\exp\left\{\int_0^{t+\Delta t} A(s)\,ds\right\} - \exp\left\{\int_0^t A(s)\,ds\right\}}{\Delta t},$$

we would like to factor, i.e.

$$\exp\left\{\int_0^t A(s)\,ds\right\} \left(\frac{\exp\left\{\int_t^{t+\Delta t} A(s)\,ds\right\} - I}{\Delta t} \right).$$

But this presumes that $e^B e^C = e^{B+C}$, but this is not true in general, since B and C may not commute.

Example: [Set-Up for Bifurcation Theory] $\ddot{y} = -\sin y$. We can change this to a system:

$$\begin{aligned} \dot{y} &= v \\ \dot{v} &= -\sin y \end{aligned}$$

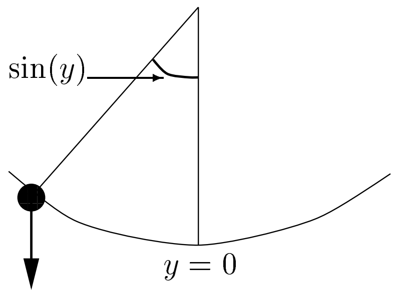

FIGURE 9. The acceleration due to gravity is of the magnitude $\sin(y)$.

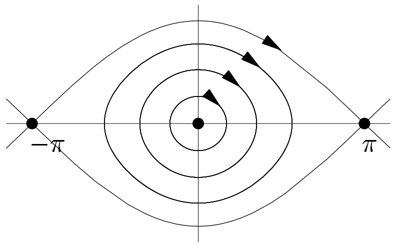

FIGURE 10. The phase plane for $\ddot{y} = -\sin(y)$

We can see that $(\pm\pi, 0)$ are hyperbolic fixed points. The point $(0,0)$ have eigenvalues $\pm i$. We have an invariant of motion $H(y,v) = v^2/2 - \cos y$.

$$\begin{aligned}\frac{d}{dt}H(y(t),v(t)) &= \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial v}\dot{v} \\ &= \sin(y)v + v(-\sin(y)) = 0\end{aligned}$$

So this is a Hamiltonian system. Add a small periodic forcing and get

$$\ddot{y} = -\sin y + \underbrace{\epsilon g(t)}_{\text{external}}$$

where $g(t+T) = g(t)$.

We can make this system autonomous:

$$\begin{aligned}\dot{y} &= v \\ \dot{v} &= -\sin(y) + \epsilon g(s) \\ \dot{s} &= 1.\end{aligned}$$

The phase space is

$$\mathbf{R}^2 \times S^1 = \{(y,v,s) | \, (y,v) \in \mathbf{R}^2, s \in S^1\}.$$

and $S^1 = [0,T]$ where 0 and T are identified.

When $\epsilon = 0$, we get

$$\begin{aligned}\dot{y} &= v \\ \dot{v} &= -\sin y \\ \dot{s} &= 1\end{aligned}$$

and the points $(0,0,0)$ and $(\pm\pi,0,0)$ correspond to periodic orbits.

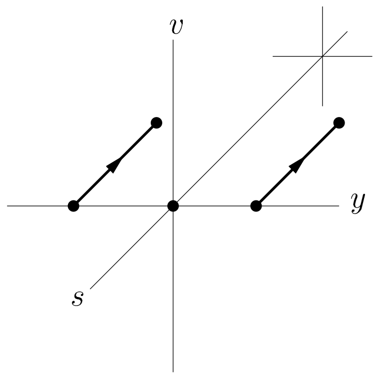

FIGURE 11. Some of the periodic orbits of the newly autonomous system

What happens to the periodic orbits when $\epsilon > 0$ but small, i.e. for small periodic forcing? Are there period T periodic orbits near $(y = 0, v = 0)$ and/or $(y = \pm\pi, v = 0)$? This is a problem in bifurcation theory: what changes or stays the same when we change a parameter?

The first step is to understand the orbits in the $\epsilon = 0$ case. Compute the Floquet multipliers of $(0,0,0), (\pm\pi,0,0)$. Let's start with $(\pm\pi,0,0)$. We expect a saddle with 1-dimensional W^s, 1-dimensional W^u.

Let $\phi((y,v,s,),t)$ be the solution flow for

$$\begin{aligned} \dot{y} &= v \\ \dot{v} &= -\sin y \qquad\qquad = F(y,v,s) \\ \dot{s} &= 1. \end{aligned}$$

Then $\phi((\pi,0,0),t) = (\pi,0,t)$ is the periodic orbit.

$$\dot{X} = DF|_{(\pi,0,t)}\, X = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} X.$$

If we diagonalize, with

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we see that

$$P^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Making a change of coordinates $PZ = X$, we get that

$$\dot{Z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} Z, \qquad Z(0) = I.$$

So

$$Z(t) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and changing back to the original coordinates we get

$$X(t) = \frac{1}{2}\begin{pmatrix} e^t & -e^{-t} & 0 \\ e^t & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

At time T the eigenvalues are $e^T, e^{-T}, 1$. Note that one is greater than 1, one is less than 1. This is a hyperbolic periodic orbit.

Now, at $(0,0,0)$, $\phi((0,0,0),t) = (0,0,t)$, and

$$\dot{X} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} X, \qquad X(0) = I.$$

Thus

$$X(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the eigenvalues of this matrix are $\cos(T) \pm \sin(T), 1$. This is not hyperbolic. If $T = 2n\pi$ then the eigenvalues are $1, 1, 1$, and if $T = (2n+1)\pi$ then the eigenvalues are $-1, -1, 1$.

4. Bifurcation Theory (Nov. 8)

Consider the family of systems $\dot{x} = F_\epsilon(x)$, where $F_\epsilon \colon \mathbf{R}^n \to \mathbf{R}^n$ for all ϵ. If $\epsilon \in \mathbf{R}$, we call this a 1-parameter family, and if $\epsilon \in \mathbf{R}^m$, we call this an m-parameter family.

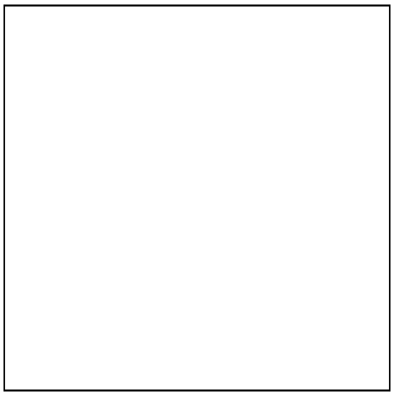

FIGURE 12

How does the solution change as ϵ changes? We say that a **bifurcation** occurs at $\epsilon = \epsilon_0$ if the flow for $\epsilon < \epsilon_0$ is "different" from the flow for $\epsilon > \epsilon_0$. It does depend on the context, however, what you mean by "different".

Example: Is the flow for

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{(See Figure 13)}$$

different from

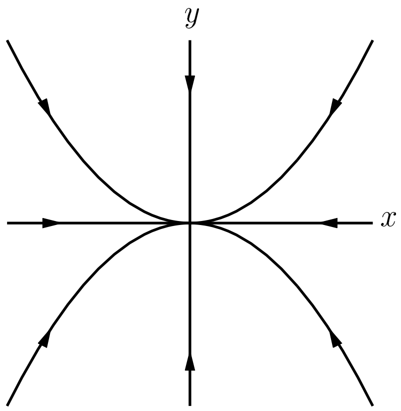

FIGURE 13. A straight sink

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} ? \text{ (See Figure 14)}$$

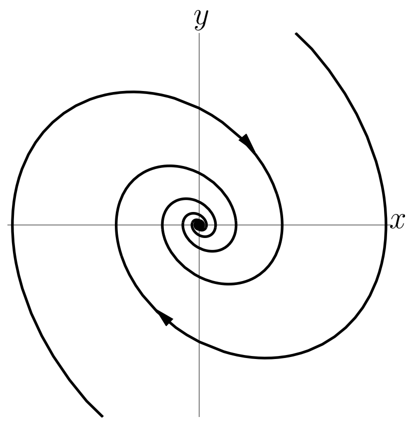

FIGURE 14. A spiral sink

It is certainly different from

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \text{ (See Figure 15)}$$

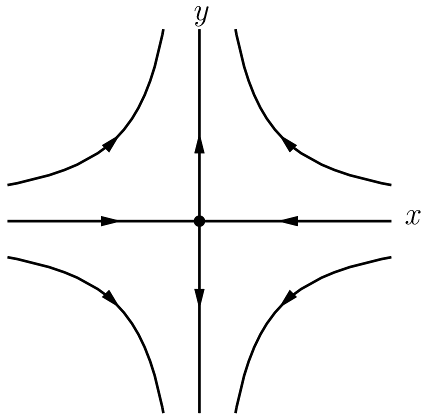

FIGURE 15. A saddle

The most important thing to note is that bifurcations almost never happen.

THEOREM 4.1 (Straightening-Out Lemma). *Given a vector field $F\colon \mathbf{R}^n \to \mathbf{R}^n$, suppose that $F(x_0) \neq 0$, then there is a neighborhood V of x_0 and coordinates on V such that*

$$F(x) = (1, 0, 0, \ldots, 0) \qquad \text{for all } x \in V$$

In these coordinates, the solution flow is $\phi(x, t) = x + (t, 0, \ldots, 0)$.

Idea of Proof.

1. Choose surface of section Σ at x_0 such that vector field is never tangent to Σ.
2. Choose coordinates so that $\Sigma \subseteq \{(0, x_2, \ldots, x_n)\}$.
3. For x close to Σ, find a small time t so that $\phi(x, t) \in \Sigma$. Then coordinates of x are $(-t, x_2, \ldots, x_n)$ where $\phi(x, t) = (0, x_2, \ldots, x_n)$.
4. Check, in these coordinates, that $F = (1, 0, \ldots, 0)$.

So, give a 1-parameter family of flows, if $F_{\epsilon_0}(x_0) \neq 0$ for some choice of ϵ_0, x_0, then (assuming that F is at least continuous in ϵ), for ϵ near ϵ_0, $F_\epsilon(x_0) \neq 0$. So there is a neighborhood about x_0 so that the flow for F_ϵ is just the straight-line flow. Up to a change of coordinates, locally, there is no change in the flow. This says that bifurcations can only happen

1. at $F_\epsilon(x_0) = 0$ (at fixed points), or
2. globally.

These global bifurcations are hard to see, need to find another way to make them “local”.

We will have three ways of attacking this problem:

1. Bifurcations of fixed points (infinitely many cases),
2. Bifurcations of periodic orbits (fixed points on Poincaré maps), and the
3. Melnikov method

So, for the first:

Let F_ϵ be a 1-parameter family, and assume that F is as smooth as necessary in x and ϵ. Assume we have a fixed point $F_0(0) = 0$. Again, we will see that bifurcations of fixed points almost never happen.

THEOREM 4.2. *If 0 is a hyperbolic fixed point of F_0, then there is an $\epsilon_1 > 0$ and*

$$\gamma\colon (-\epsilon_1, \epsilon_1) \to \mathbf{R}^n$$

satisfying

1. $\gamma(0) = 0$,
2. $F_\epsilon(\gamma(\epsilon)) = 0$,
3. $\gamma(\epsilon)$ *is smooth (as smooth as* F_ϵ,
4. *there exists a neighborhood* V *of* 0 *such that if* $\epsilon \in (-\epsilon_1, \epsilon_1)$ *and* $x \in V$, *and* $F_\epsilon(x) = 0$, *then* $\gamma(\epsilon) = x$.
5. *The dimensions of* $W^s_{F_\epsilon}(\gamma(\epsilon))$ *and* $W^u_{F_\epsilon}(\gamma(\epsilon))$ *are the same as* $W^s_{F_0}(0)$ *and* $W^u_{F_0}(0)$ *(and* $\gamma(\epsilon)$ *is hyperbolic for* F_ϵ*).*

Also, $D_x F_\epsilon(\gamma(\epsilon))$ *is continuous in* ϵ.

Proof: Let $\mathcal{F}\colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$, with $\mathcal{F}(x,\epsilon) = F_\epsilon(x)$. We know that $\mathcal{F}(0,0) = 0$. Location of fixed points is given by 0 level set of $\mathcal{F}$. What can we say about $\mathcal{F}^{-1}(0)$?

The Implicit Function Theorem tells us that there is an ϵ and a smooth function

$$\gamma\colon (-\epsilon_1, \epsilon_1) \to \mathbf{R}^2$$

so that

$$\mathcal{F}(\gamma(\epsilon), \epsilon) = (0,0),$$

i.e., the level set is a graph of γ.

The hypothesis that has to be satisfied is some condition on the partials of $\mathcal{F}$, This might not hold if the conditions of the theorem do not hold.

Compute $\dfrac{d\gamma}{d\epsilon}$ where $\gamma(\epsilon) = (\gamma_1(\epsilon), \gamma_2(\epsilon))$. We know that

$$\mathcal{F}\left(\gamma_1(\epsilon), \gamma_2(\epsilon)\right), \epsilon = (0,0).$$

Let $F_\epsilon(x) = (f(x), g(x))$. Then we need

$$\frac{d}{d\epsilon}(\mathcal{F}(\gamma_1(\epsilon), \gamma_2(\epsilon), \epsilon) = \frac{d}{d\epsilon} F_\epsilon(0,0) = (0,0).$$

Computing, we need:

$$\begin{pmatrix} \dfrac{\partial f_\epsilon}{\partial x_1} & \dfrac{\partial f_\epsilon}{\partial x_2} \\ \dfrac{\partial g_\epsilon}{\partial x_1} & \dfrac{\partial g_\epsilon}{\partial x_2} \end{pmatrix} \begin{pmatrix} \dfrac{d\gamma_1}{d\epsilon} \\ \dfrac{d\gamma_2}{d\epsilon} \end{pmatrix} + \begin{pmatrix} \dfrac{df_\epsilon}{d\epsilon} \\ \dfrac{dg_\epsilon}{d\epsilon} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So we need

$$\begin{pmatrix} \dfrac{\partial f_\epsilon}{\partial x_1} & \dfrac{\partial f_\epsilon}{\partial x_2} \\ \dfrac{\partial g_\epsilon}{\partial x_1} & \dfrac{\partial g_\epsilon}{\partial x_2} \end{pmatrix}_{x=0,\epsilon=0} = D_x F_0|_{x=0}$$

to be hyperbolic.

No zero eigenvalues, so $DF_0|_{x=0}$ exists. ∎

5. Hyperbolicity in bifurcations (Nov. 11)

Recall the proof of the last theorem:

$$\mathcal{F}\colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$$
$$(x, \epsilon) \mapsto F_\epsilon(x)$$

with the fixed points the level set $\mathcal{F}^{-1}(0)$. The Inverse Function Theorem gives us that this level set is the graph of a function. If we differentiate $\mathcal{F}(\gamma(\epsilon), \epsilon)$ with respect to ϵ, we get:

$$D_x\mathcal{F} \cdot \frac{d\gamma}{d\epsilon} + D_\epsilon \mathcal{F} = 0,$$

so

$$\frac{d\gamma}{d\epsilon} = -\left(D_x\mathcal{F}\right)^{-1} \cdot D_\epsilon\mathcal{F}.$$

We know that $D_x\mathcal{F}$ is invertible because 0 is hyperbolic. To get information about $\gamma(\epsilon)$, we compute the Taylor series. We need to compare the flow near $\gamma(\epsilon)$ for the solution flow of $\dot{x} = F_\epsilon(x)$ to flow near 0 for $F_0(x)$.

Compute the linear part:

$$\left.(D_x F_\epsilon)\right|_{\gamma(\epsilon)}$$

and this gives a curve of matrices

$$M(\epsilon) = \left.(D_x F_\epsilon)\right|_{x=\gamma(\epsilon)}.$$

As we change ϵ, components change smoothly with ϵ, which means the eigenvalues change continuously with ϵ.

Example: For the matrix

$$\begin{pmatrix} -1 & \epsilon \\ -\epsilon & 1 \end{pmatrix},$$

we can compute the eigenvalues, and see that they are $-1 \pm i\,|\epsilon|$. So the roots of the polynomial are only continuous in the coefficients, and not necessarily smooth.

So there is an interval where the number of eigenvalues with real part positive, and the number with real part negative, is constant for $M(\epsilon)$. However, we know that $W^s_{F_\epsilon}(\gamma(\epsilon))$ and $W^u_{F_\epsilon}(\gamma(\epsilon))$ change smoothly with ϵ.

This leads to a

Scholium. In the theorem, we only used that $\left.D_x F_0\right|_0$ was invertible to get a curve $\gamma(\epsilon)$, i.e. no eigenvalues $= 0$. Thus we need only invertibility, not hyperbolicity, to apply the Inverse Function Theorem. However, if we do not assume hyperbolicity, the dimensions of the unstable and stable manifolds can change. Bifurcations can happen at equilibrium points where eigenvalues are 0 or are pure imaginary.

Example: $F_\epsilon\colon \mathbf{R}^2 \to \mathbf{R}^2$, $F(0) = 0$,

$$\left.Df\right|_0 = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$$

Assume that $\lambda < 0$, and let $F_\epsilon(x) = (f_\epsilon(x), g_\epsilon(x))$. What happens when we change ϵ? Avoid the question of the behavior of the solution of $\dot{x} = F_0(x)$ for as long as possible. Try the same trick:

$$\mathcal{F}(x, \epsilon) = F_\epsilon(x)$$

We know that $D_x\mathcal{F}^{-1}$ does not exist at $x = 0, \epsilon = 0$. But we're still looking for $\mathcal{F}^{-1}(\{(0,0)\})$. Try to look at level set as a graph in another direction:

Let $x = (x_1, x_2)$, Try to find $(\eta(x_1), \zeta(x_1))$ such that

$$\mathcal{F}(x_1, \eta(x_1), \zeta(x_1)) = (0,0),$$

i.e.

$$\begin{pmatrix} f_{\zeta(x_1)}(x_1, \eta(x_1)) \\ g_{\zeta(x_1)}(x_1, \eta(x_1)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

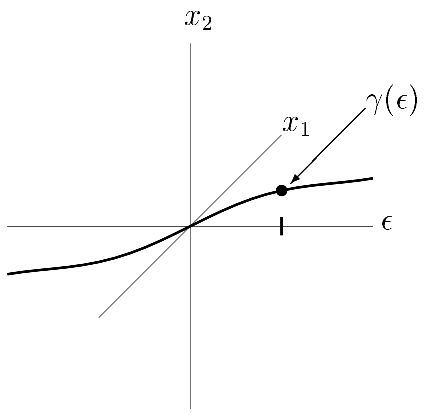

FIGURE 16. $\gamma(\epsilon)$ is a graph

Let's try to compute $\dfrac{d\eta}{dx_1}, \dfrac{d\zeta}{dx_1}$:

$$\begin{pmatrix} \dfrac{\partial f}{\partial x_1} + \dfrac{\partial f}{\partial x_2}\dfrac{\partial \eta}{\partial x_1} + \dfrac{\partial f}{\partial \epsilon}\dfrac{\partial \zeta}{\partial x_1} \\ \dfrac{\partial g}{\partial x_1} + \dfrac{\partial g}{\partial x_2}\dfrac{\partial \eta}{\partial x_1} + \dfrac{\partial g}{\partial \epsilon}\dfrac{\partial \zeta}{\partial x_1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so we have to have

$$\begin{pmatrix} \dfrac{\partial \eta}{\partial x_1} \\ \dfrac{\partial \zeta}{\partial x_1} \end{pmatrix} = -\begin{pmatrix} \dfrac{\partial f}{\partial x_2} & \dfrac{\partial f}{\partial \epsilon} \\ \dfrac{\partial g}{\partial x_2} & \dfrac{\partial g}{\partial \epsilon} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial g}{\partial x_1} \end{pmatrix}.$$

Thus we need

$$\begin{pmatrix} \dfrac{\partial f}{\partial x_2} & \dfrac{\partial f}{\partial \epsilon} \\ \dfrac{\partial g}{\partial x_2} & \dfrac{\partial g}{\partial \epsilon} \end{pmatrix}_{x=0,\epsilon=0}$$

to be invertible.

We know that $\left.\dfrac{\partial f}{\partial x_2}\right|_{(0,0)} = 0$ and $\left.\dfrac{\partial g}{\partial x_2}\right|_{(0,0)} = \lambda$.

This gives us

$$\begin{pmatrix} 0 & \dfrac{\partial f}{\partial \epsilon} \\ \lambda & \dfrac{\partial g}{\partial \epsilon} \end{pmatrix}_{x=0,\epsilon=0}$$

which is clearly invertible iff $\left.\dfrac{\partial f}{\partial \epsilon}\right|_{(0,0)} \neq 0$.

This means that the x_1-component of F_0 depends on ϵ, i.e. changes at nonzero speed as ϵ changes.

THEOREM 5.1. *Given* $F_\epsilon \colon \mathbf{R}^2 \to \mathbf{R}^2$, $F_0(0) = 0$,

$$D_x F_0|_{x=0} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}, \qquad \lambda < 0,$$

provided that

$$\frac{\partial\{\textit{1st component of } F\}}{\partial \epsilon}_{(x=0,\epsilon=0)} \neq 0,$$

then there is a $\delta_1 > 0$ *and a curve* $\xi \colon (-\delta_1, \delta_1) \to \mathbf{R}^2$, $\xi(x_1) = (\eta(x_1), \zeta(x_1))$, *and a neighborhood* V *of* $((0,0),0)$ *such that*

1. $F_{\zeta(x_1)}(x_1, \eta(x_1)) = 0$ *for all* $x \in (-\delta_1, \delta_1)$,
2. *If* $((x_1, x_2), \epsilon) \in V$ *and* $F_\epsilon(x_1, x_2) = 0$ *then* $x_2 = \eta(x_1)$, $\epsilon = \zeta(x_1)$, *and*
3. $\xi = (\eta, \zeta)$ *is smooth.*

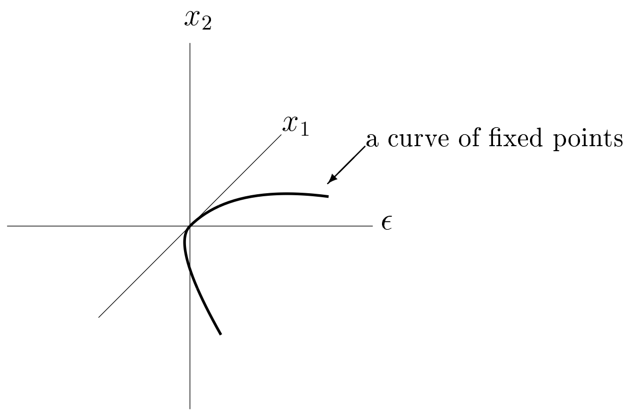

FIGURE 17. We can see that there are no fixed points for $\epsilon < 0$, and two for $\epsilon > 0$.

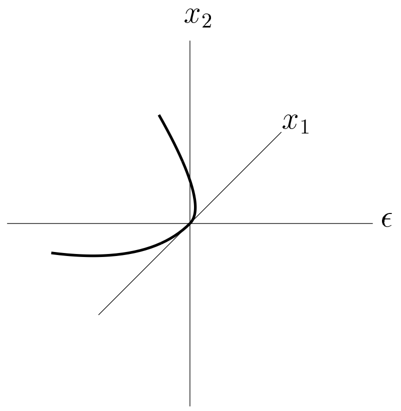

FIGURE 18. Here, we have two fixed points for $\epsilon < 0$, and none for $\epsilon > 0$. The difference between this picture and the last is the sign of the second partial.

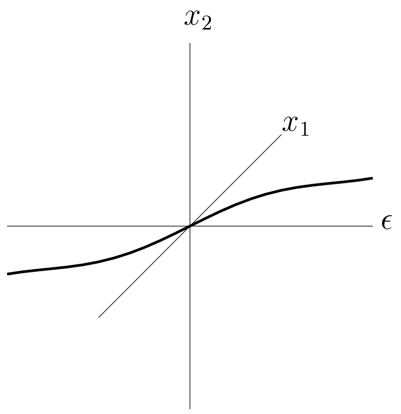

FIGURE 19. No bifurcation here.

Need to compute more about $\xi(x_1)$, and we need its 1st and 2nd derivatives. We know from before that

$$\begin{pmatrix} \dfrac{\partial \eta}{\partial x_1} \\ \dfrac{\partial \zeta}{\partial x_1} \end{pmatrix} = -\begin{pmatrix} \dfrac{\partial f}{\partial x_2} & \dfrac{\partial f}{\partial \epsilon} \\ \dfrac{\partial g}{\partial x_2} & \dfrac{\partial g}{\partial \epsilon} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial g}{\partial x_1} \end{pmatrix}$$
$$= \frac{1}{\dfrac{\partial f}{\partial x_1}\dfrac{\partial g}{\partial \epsilon} - \dfrac{\partial f}{\partial \epsilon}\dfrac{\partial g}{\partial x_2}} \begin{pmatrix} \dfrac{\partial g}{\partial \epsilon}\dfrac{\partial f}{\partial x_1} - \dfrac{\partial f}{\partial \epsilon}\dfrac{\partial g}{\partial x_1} \\ -\dfrac{\partial g}{\partial x_2}\dfrac{\partial f}{\partial x_1} + \dfrac{\partial f}{\partial x_2}\dfrac{\partial g}{\partial x_1} \end{pmatrix}.$$

At $x = 0, \epsilon = 0$, we get $(0,0)$, i.e. the curve $(x_1, \eta(x_1), \zeta(x_1))$ is tangent to the x_1-axis at $(0,0)$. We also have

$$\frac{d^2\zeta}{dx_1^2} = \frac{-\partial^2 f_\epsilon/\partial x_1^2}{\partial f_\epsilon/\partial\epsilon}, \quad \frac{d^2\eta}{dx_1^2} = 0.$$

6. Bifurcation diagrams (Nov. 13)

We want to know if the nondegeneracy conditions from the last chapter are satisfied, so let's try to reduce dimension, since nothing much happens in the x_2-direction.

Consider

$$\dot{\begin{pmatrix} x_1 \\ x_2 \\ \epsilon \end{pmatrix}} = \begin{pmatrix} F_\epsilon(x_1) \\ F_\epsilon(x_2) \\ 0 \end{pmatrix} = \hat{F}\begin{pmatrix} x_1 \\ x_2 \\ \epsilon \end{pmatrix}.$$

At the fixed point $(0,0,0)$,

$$D_{(x_1,x_2,\epsilon)}F\big|_{(0,0,0)} = \begin{pmatrix} 0 & 0 & \partial f_\epsilon/\partial\epsilon \\ 0 & \lambda & \partial g_\epsilon/\partial\epsilon \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues are $0, 0$, and λ. So there is a 2-dimensional center manifold and a 1-dimensional (since $\lambda < 0$) strong stable manifold, W^{ss}.

Note: All fixed points must be on $W^c(0,0,0)$, since if they are not, in backwards time, the distances stretch exponentially.

Consider the flow restricted to $W^c(0,0,0)$, and return to the 1-parameter family. If ϵ is considered as a coordinate on $W^c(0,0,0)$, then $\dot\epsilon = 0$, so we have

$$\dot{x} = h_\epsilon(x),$$

a 1-dimensional, 1-parameter family.

We know that $H_0(0) = 0$, and $\frac{dh}{dx}(0,0) = 0$, and we require that

$$\frac{\partial h}{\partial\epsilon} \neq 0 \qquad \text{and} \qquad \frac{\partial^2 h}{\partial x^2} \neq 0.$$

Bifurcation diagrams:

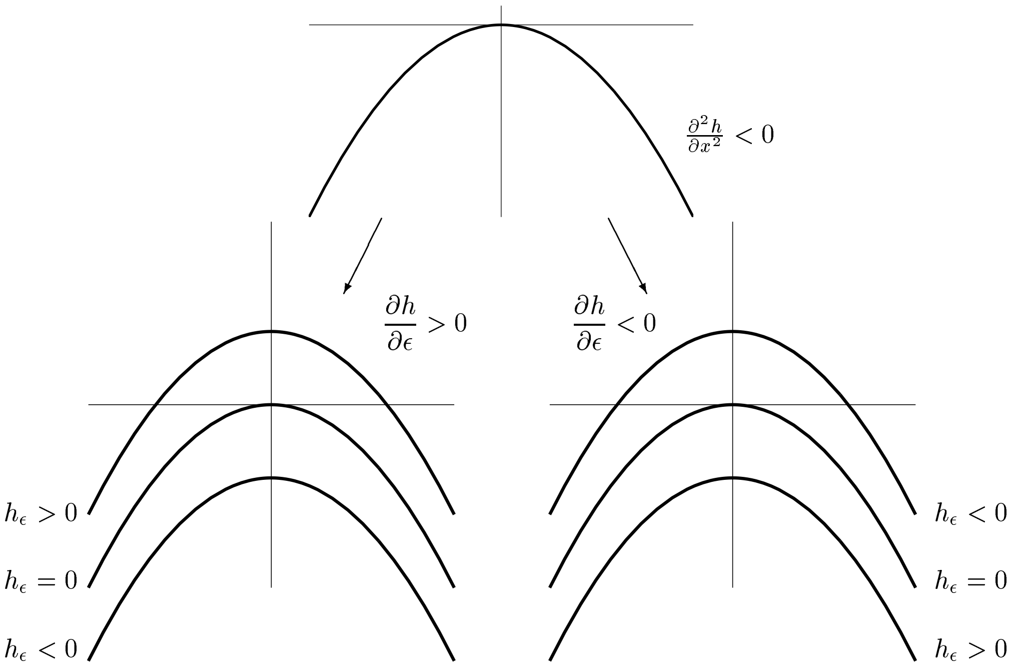

FIGURE 20. These are the possible bifurcations if the second partial is negative. The difference between the two bottom pictures is the sign of the first partial $\dfrac{\partial h}{\partial \epsilon}$. When it is negative, for example, we go from no fixed points to two.

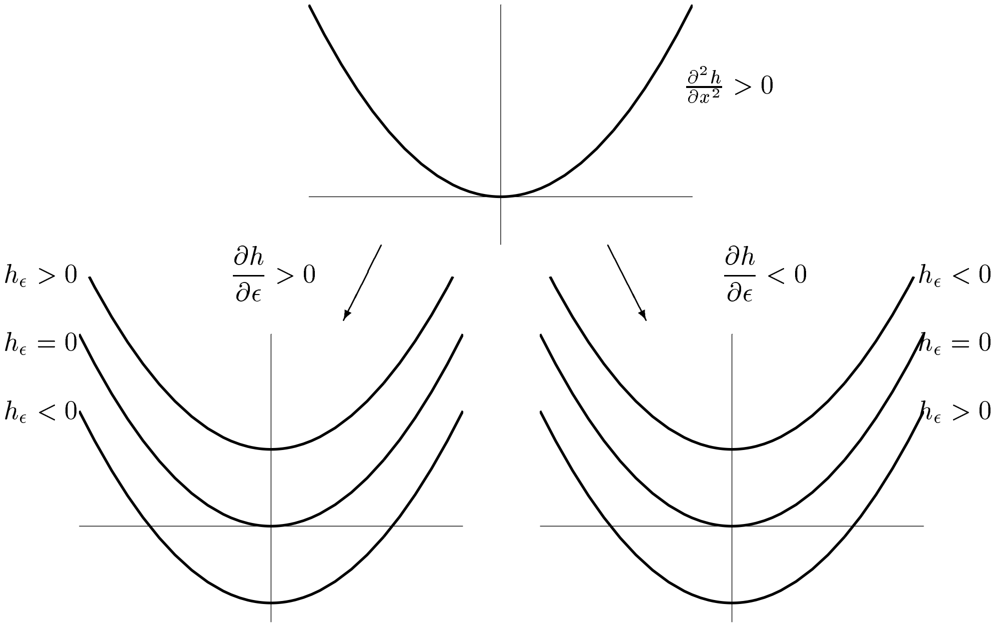

FIGURE 21. These are the possible bifurcations if the second partial is positive. See the last figure also. Again, the sign of the first partial determines whether we go from two fixed points to none or vice-versa.

A **saddle-node bifurcation** occurs when there is 1 simple zero eigenvalue with nondegeneracy conditions. What about 2 zero eigenvalues, such as

$$DF_0|_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{or even} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}?$$

And why would we consider such a degenerate situation?

So, to study these situations, we need a 2-parameter family. (See Figure 25

7. Spiral sinks and spiral sources, normal form calculation (Nov. 15)

Consider $F_\epsilon \colon \mathbf{R}^n \to \mathbf{R}^n$, $F_0(0) = 0$, and $DF_0|_0$ has eigenvalues $\pm i\omega, \lambda_3, \ldots, \lambda_n$, with the real part of the λ_j all not 0.

First, restrict attention to 2 dimensions, and consider $F_\epsilon \colon \mathbf{R}^2 \to \mathbf{R}^2$. We can justify this by saying that if the dimension is > 2 then restrict to the center manifold for

$$\hat{F}\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \epsilon \end{pmatrix} = \begin{pmatrix} F_\epsilon(x_1) \\ \vdots \\ F_\epsilon(x_n) \\ 0 \end{pmatrix}$$

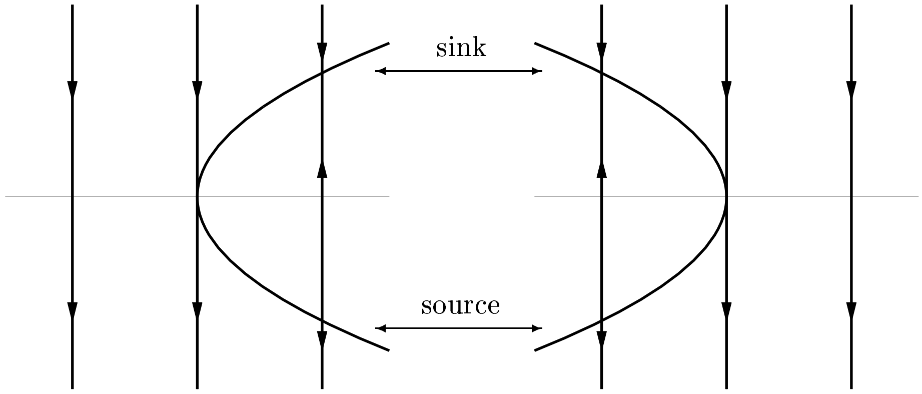

FIGURE 22. In the left picture, we undergo a bifurcation from no fixed points, through a node, and then to a sink-source pair. The parabola represents the fixed points. Any vertical slice through the picture corresponds to the system at some value of ϵ. The picture on the right is the same, except in reverse order.

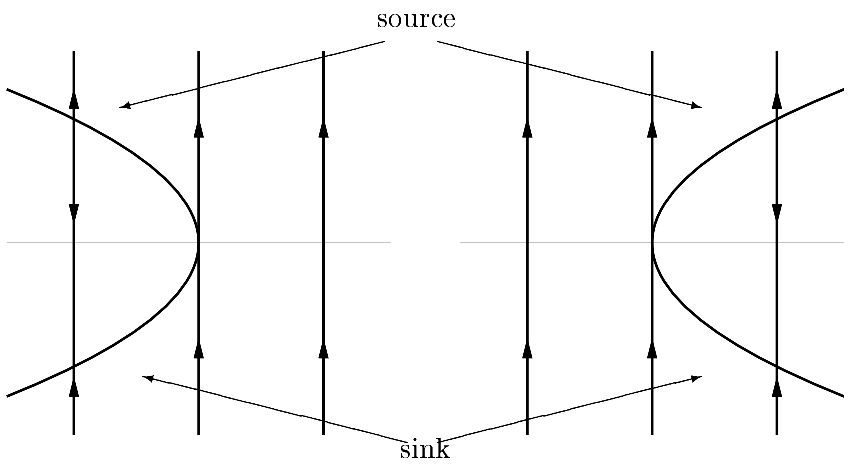

FIGURE 23. In the left picture, we undergo a bifurcation from no fixed points, through a node, and then to a sink-source pair. The parabola represents the fixed points. Any vertical slice through the picture corresponds to the system at some value of ϵ. The picture on the right is the same, except in reverse order.

which is 3-dimensional and one dimension is a parameter. We know that there is a curve $\gamma(\epsilon)$ such that $F_\epsilon(\gamma(\epsilon)) = 0$. Do an ϵ-dependent change of variables to get

$$G_\epsilon(x) = F_\epsilon(x + \gamma(\epsilon)).$$

So $G_\epsilon(0) = 0$ for all ϵ, so we can assume that $\epsilon = 0$, and that the fixed point is at 0.

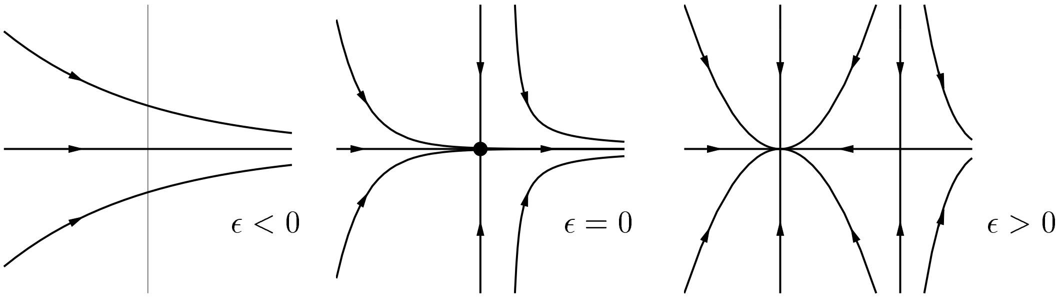

FIGURE 24. This is a saddle-node bifurcation. In this figure, we see that there are no fixed points for $\epsilon < 0$. At $\epsilon = 0$ is the bifurcation, and we have a node. For positive ϵ, we have the birth of a sink and a saddle.

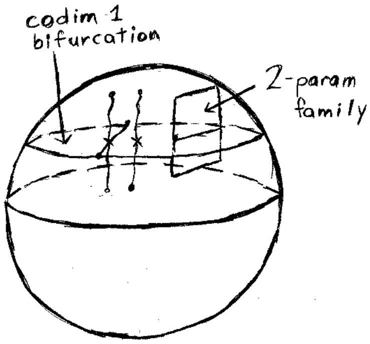

FIGURE 25. You can't get around the two-parameter family.

So we can assume that the system looks like

$$DF_0|_0 = \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}.$$

We then expect

1. Higher order terms of F_0 make 0 into a spiral sink or a spiral source.
2. If we change ϵ, we expect the eigenvalues of $DF_\epsilon|_0$ to have nonzero real part

The above conditions will be our "nondegeneracy conditions" which insure that there is a bifurcation. Suppose, for example, the higher order terms of F_0 make 0 a spiral sink, so there must exist an attractor block for this sink.

Suppose we change ϵ a little and eigenvalues of DF_0 have positive real part. Then 0 is a spiral source (hyperbolic). But old attractor block for $\epsilon = 0$ is still an attractor block.

Orbits entering the block must limit somewhere, but it can't be 0, and there are no other equilibria.

THEOREM 7.1 (Poincaré-Bendixon). *A bounded orbit of a flow in* $\mathbf{R}^2$ *limits onto a periodic orbit or a fixed point or a "limit cycle".*

So there must be at least one periodic orbit near zero. The point of the Hopf Bifurcation Theorem is to show

1. that there is only 1 periodic orbit, and
2. to give some machinery for manipulating higher order terms so we can see for which ϵ's this orbit exists.

So, as above, we assume that

$$DF_0|_0 = \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}.$$

We want to try to get a handle on the higher order terms of F_ϵ.

$$\begin{aligned} F_0 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} ax^2 + bxy + cy^2 \\ \alpha x^2 + \beta xy + \gamma y^2 \end{pmatrix} + \text{ h.o.t.} \\ &= A \begin{pmatrix} x \\ y \end{pmatrix} + \big(\text{2nd order}\big) + \text{ h.o.t.} \end{aligned}$$

If 0 is a spiral sink, then a change of variables won't change that. So what is it about power series expansion that is topological information?

Try to simplify the power series of F_0 by changing variables. Is there any topological information in the 2nd order terms, i.e. can I do a change of variables so that in the new variables,

$$\dot{\begin{pmatrix} z \\ w \end{pmatrix}} = A \begin{pmatrix} z \\ w \end{pmatrix} + \begin{pmatrix} \text{3rd order} \\ \text{and higher} \end{pmatrix}?$$

So, let's try a change of variables of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix} + h \begin{pmatrix} z \\ w \end{pmatrix},$$

where

$$h \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} a_1 z^2 + b_1 zw + c_1 w^2 \\ \alpha_1 z^2 + \beta_1 zw + \gamma_1 w^2 \end{pmatrix},$$

i.e. identity plus 2nd order terms, and we can pick $a_1, b_1, \ldots, \gamma_1$. Now,

$$F_0 \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + F_2 \begin{pmatrix} x \\ y \end{pmatrix} + \text{ h.o.t.}$$

Then

$$\begin{aligned} \dot{\begin{pmatrix} x \\ y \end{pmatrix}} &= \dot{\begin{pmatrix} z \\ w \end{pmatrix}} + Dh|_{z,w} \dot{\begin{pmatrix} z \\ w \end{pmatrix}} \\ &= F \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + F_2 \begin{pmatrix} x \\ y \end{pmatrix} + \text{ h.o.t.} \\ &= A \left(\begin{pmatrix} z \\ w \end{pmatrix} + h \begin{pmatrix} z \\ w \end{pmatrix} \right) + F_2 \begin{pmatrix} z \\ w \end{pmatrix} + \text{ h.o.t.} \end{aligned}$$

Thus we have that

$$\left(I + Dh|_{z,w}\right) \dot{\begin{pmatrix} z \\ w \end{pmatrix}} = A \begin{pmatrix} z \\ w \end{pmatrix} + Ah \begin{pmatrix} z \\ w \end{pmatrix} + F_2 \begin{pmatrix} z \\ w \end{pmatrix} + \text{ h.o.t.},$$

or

$$\dot{\begin{pmatrix} z \\ w \end{pmatrix}} = \left(I + Dh|_{z,w}\right)^{-1} \left(A\begin{pmatrix} z \\ w \end{pmatrix} + Ah\begin{pmatrix} z \\ w \end{pmatrix} + F_2\begin{pmatrix} z \\ w \end{pmatrix} + \text{ h.o.t.}\right).$$

Can we write $(I+Dh)^{-1}$ as $I+B$? Well,

$$I = (I+Dh)\cdot(I+B) = I + Dh + B + Dh\cdot B,$$

so, up to first order, $(I+Dh)^{-1} = (I-Dh)$. Thus

$$\dot{\begin{pmatrix} z \\ w \end{pmatrix}} = \left(I - Dh|_{z,w}\right) \left(A\begin{pmatrix} z \\ w \end{pmatrix} + Ah\begin{pmatrix} z \\ w \end{pmatrix} + F_2\begin{pmatrix} z \\ w \end{pmatrix} + \text{ h.o.t.}\right).$$

Since

$$Dh|_{z,w} = \begin{pmatrix} 2a_1 z + b_1 w & b_1 z + 2c_1 w \\ 2\alpha_1 z + \beta_1 w & \beta_1 z + 2\gamma_1 w \end{pmatrix},$$

$$\dot{\begin{pmatrix} z \\ w \end{pmatrix}} = \begin{pmatrix} 1 - 2a_1 z + b_1 w & -b_1 z - 2c_1 w \\ -2\alpha_1 z - \beta_1 w & 1 - \beta_1 z - 2\gamma_1 w \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} \eta w + (\eta\alpha_1 + a_1 - b_1\eta)z^2 + (-2a_1\eta + \eta\beta_1 + b - c_1\eta)zw + (-b_1\eta + \eta\gamma_1 + c)w^2 \\ -\eta t + (\beta_1\eta - \eta a_1 + \alpha)z^2 + (2\alpha_1\eta + 2\gamma_1\eta - \eta b_1 + \beta)zw + (\beta_1\eta - \eta c_1 + \gamma)w^2 \end{pmatrix}.$$

Make all coefficients 0,

$$\begin{pmatrix} 0 & 0 & 0 & \eta & 0 & -\eta \\ -2\eta & 0 & -2\eta & 0 & \eta & 0 \\ 0 & -\eta & 0 & 0 & 0 & \eta \\ -\eta & 0 & 0 & 0 & \eta & 0 \\ 0 & -\eta & 0 & 2\eta & 0 & 2\eta \\ 0 & 0 & -\eta & 0 & -\eta & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ -c \\ -\alpha \\ -\beta \\ -\gamma \end{pmatrix}$$

The determinant of this matrix is $-3\eta^6 \neq 0$, so we can change variables and eliminate all 2nd order terms.

Remark: This is an example of a normal form calculation. We will try the cubic terms next.

8. More normal form calculations, complexification (Nov. 18)

In the case we have been talking about, the linear part is a center. But we could have a change in $\dim W^u, W^s$ at $\epsilon = 0$. What is happening at $\epsilon = 0$?

Let $\lambda_\epsilon, \overline{\lambda_\epsilon}$ by the eigenvalues of $DF_\epsilon|_0$. So $\lambda_\epsilon = \mu_\epsilon + i\eta_\epsilon$ where $\mu_0 = 0, \eta_0 = \eta$. Write

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = F_\epsilon\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu_\epsilon & \eta_\epsilon \\ \eta_\epsilon & \mu_\epsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \text{ h.o.t.}$$

In the last section, we saw that we can do an algebraic change of variables which eliminates all 2nd order terms of F_0. How many terms can we eliminate? Which terms have geometric information? Let us take advantage of the fact that we're in $\mathbf{R}^2$, and use complex-valued algebra.

Think of $(x,y) \in \mathbf{C}^2$, and can linearize linear part. Let

$$P = \begin{pmatrix} 1 & 1 \\ i & i \end{pmatrix} \text{ for } \begin{pmatrix} \mu_\epsilon & \eta_\epsilon \\ \eta_\epsilon & \mu_\epsilon \end{pmatrix}.$$

Do change of variables

$$P\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

and get the system

$$\begin{aligned} \dot{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}} &= P^{-1}\begin{pmatrix} \mu_\epsilon & \eta_\epsilon \\ \eta_\epsilon & \mu_\epsilon \end{pmatrix} P \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \text{ h.o.t.} \\ &= \begin{pmatrix} \lambda_\epsilon & 0 \\ 0 & \lambda_\epsilon \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \text{ h.o.t.} \end{aligned}$$

Where did the original $\mathbf{R}^2$ go, i.e. where is $\{(x,y)|\ x,y \in \mathbf{R}\}$? Since

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix},$$

so

$$\begin{aligned} z_1 &= (x - iy)/2 \\ z_2 &= (x + iy)/2. \end{aligned}$$

Let $\Delta = \{(z_1, z_2)|\ z_2 = \overline{z}_1\}$, a 2-dimensional subspace of the 4-dimensional $\mathbf{C}^2$. Moreover, since the space $\{(x,y)|\ x,t \in \mathbf{R}\}$ is invariant in the original equation, Δ is invariant under the transformed equation. So if we only care about $(x,y) \in \mathbf{R}$, then we need only consider solutions on Δ where $z_2 = \overline{z}_1$.

So we take the expansion

$$\dot{z}_1 = \lambda_\epsilon z_1 + F_2(z_1, z_2)$$

and replace z_2 with $\overline{z}_1$. So

$$\overline{z_1} = \lambda_\epsilon z_1 + \sum_{i+j\geq 2} a_{ij} z_1^i \overline{z}_1^j.$$

(If F is $\mathcal{C}^{N+1}$, then

$$\overline{z_1} = \lambda_\epsilon z_1 + \sum_{i+j\geq 2} a_{ij} z_1^i \overline{z}_1^j + \text{terms of order } N+1.)$$

Remarks:

1. This is not complex analytic!
2. a_{ij} also depend on ϵ.

Now, try to do a change of variables which eliminate terms of the power series. For example, try to get rid of $z^k\overline{z}^l \quad (k+l \geq 2)$. Try a new variable $w \in \mathbf{C}$ where

$$z = w + bz^k\overline{z}^l.$$

We see that

$$\dot{z} = \dot{w} + bk\dot{w}w^{k-1}\overline{w}^l + bl\overline{w}^{l-1}w^k,$$

and

$$\lambda_\epsilon z + \sum_{i+j\geq 2} a_{ij} z^i \overline{z}^j$$
$$= \lambda_\epsilon (w + bw^k \overline{w}^l) + \sum_{i+j\geq 2} a_{ij} (w + bw^k \overline{w}^l)^i \overline{(w + bw^k \overline{w}^l)}^j.$$

So we get

$$\dot{w} = \lambda_\epsilon w + \sum_{\substack{k+l\geq i+j\geq 2 \\ i,j\neq k,l}} a_{ij} w^i \overline{z}^j$$
$$+ (\lambda_\epsilon b - bk\lambda_\epsilon - bl\overline{\lambda}_\epsilon + a_{kl}) w^k \overline{w}^l + \text{ h.o.t.}$$

All we need is

$$\lambda_\epsilon b - bk\lambda_\epsilon - bl\overline{\lambda}_\epsilon + a_{kl} = 0.$$

So let

$$b = \frac{-a_{kl}}{\lambda_\epsilon - k\lambda_\epsilon - l\overline{\lambda}_\epsilon}$$

and we need worry only if

$$\lambda_\epsilon - k\lambda_\epsilon - l\overline{\lambda}_\epsilon = 0$$

Remark: This change of variable only affects k, l terms and terms of higher order. This does change the coefficients of the higher order terms, so we must do it in order, from lowest order to highest order.

$\lambda_\epsilon = \mu_\epsilon + i\eta_\epsilon$, and there are two cases:

1. $\mu_\epsilon \neq 0, \eta_\epsilon \neq 0$. We only need that

$$\mu_\epsilon - k\mu_\epsilon - l\mu_\epsilon \neq 0 \text{ or}$$
$$\eta_\epsilon - k\eta_\epsilon + l\eta_\epsilon \neq 0$$

Since we have that $\mu_\epsilon \neq 0, \eta_\epsilon \neq 0$, we need that

$$1 - k - l \neq 0 \text{ or}$$
$$1 - k + l \neq 0.$$

Since $k + l \geq 2$, both are never 0. Thus the denominator is never 0!. So we can eliminate any non-linear terms. Does this mean that we can linearize, i.e. change variables to $\dot{w} = \lambda_\epsilon w$?

No. There are infinitely many changes of variable and the domain may shrink to 0. But there does exist a neighborhood of 0 and a polynomial change of variables such that in the new variables

$$\dot{w} = \lambda_\epsilon w + \text{ order} \geq N$$

where F is $\mathcal{C}^N$.

We can linearize formally, though (if F is $\mathcal{C}^\infty, \mathcal{C}^\omega$). This was Poincaré's thesis.

2. When $\epsilon = 0$, $\lambda_\epsilon = i\eta_\epsilon = i\eta$.

The denominator $i\eta - ki\eta + li\eta = 0$ if $1 - k + l = 0$ or $l = k - 1$. So we can eliminate all terms, except $z^2\overline{z}, z^3\overline{z}^2, z^4\overline{z}^3, \ldots$

Formally, we can change variables to

$$\dot{z} = i\eta z + \sum_{j\geq 2} a_j z^j \overline{z}^{j-1}$$

provided F is $\mathcal{C}^N$, there is a neighborhood of 0 and a change of variables such that

$$\dot{z} = i\eta z + \sum_{j\geq 2}^{j+1<2N} a_j z^j \overline{z}^{j-1} + \text{ order } N.$$

This is known as "normal form". What's the geometry in this case?
Let

$$\dot{z} = i\eta z + a_2 z^2 \overline{z} + \text{ order} \geq 4.$$

and assume $a_2 = \beta + i\gamma \neq 0$. Change to polar coordinates:

$$\begin{aligned}\dot{z} &= \dot{r}e^{i\theta} + ir\dot{\theta}e^{i\theta}\\ &= i\eta r e^{i\theta} + (\beta + i\gamma) r^3 e^{i\theta} + \text{ h.o.t.}\end{aligned}$$

So

$$\begin{aligned}\dot{r} &= \beta r^3 + \text{ h.o.t. } \geq 4\\ \dot{\theta} &= \eta + \gamma r^2 + \text{ h.o.t. } \geq 3.\end{aligned}$$

If $\beta > 0$, we have a spiral source, as in Figure 26.

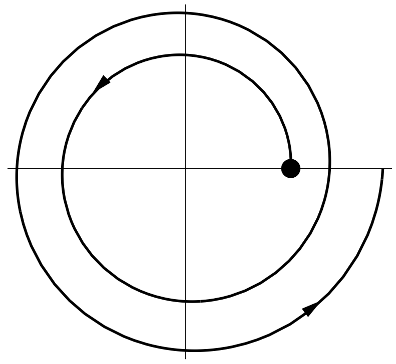

FIGURE 26. The nonlinear part makes this a spiral source.

If $\beta < 0$, we have a spiral sink,as in Figure 27.
If $\beta = 0$, we still have no info, so have to look at $z^3\overline{z}^2$ terms.
Remark: What does γ tell you? If $\gamma \neq 0$, the amount of rotation changes with r.

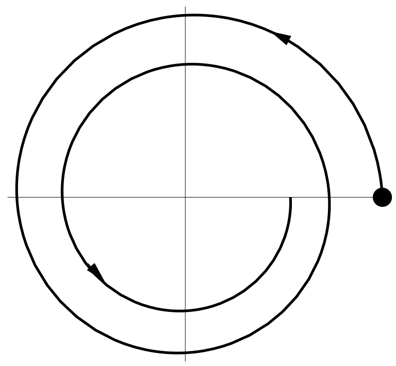

Figure 27. The nonlinear part makes this a spiral source.

CHAPTER 4

Topics

1. Setup for Hopf Bifurcation Theorem (Nov. 20)

We know from before the the eigenvalues of $DF_0|_0$ are $\pm i\eta$. Let $\lambda_0 = i\eta$, and put $\dot{z} = F_\epsilon(z)$ into normal form near $\epsilon = 0$, i.e.

$$\dot{z} = \lambda_\epsilon(z) + a_1(\epsilon)z^2\overline{z} + \text{h.o.t.}$$

Take $\lambda_\epsilon = \mu_\epsilon + i\eta_\epsilon$, and $a_1(\epsilon) = \beta_\epsilon + i\gamma_\epsilon$. In polar coordinates, we get

$$\dot{r}e^{i\theta} + i\dot{\theta}re^{i\theta} = (\mu_\epsilon + i\eta_\epsilon)re^{i\theta} + (\beta_\epsilon + i\gamma_\epsilon)r^3e^{i\theta} + \text{h.o.t.}$$

so

$$\begin{aligned}\dot{r} &= \mu_\epsilon r + \beta_\epsilon r^3 + \text{h.o.t.}\\ \dot{\theta} &= \eta_\epsilon + \gamma_\epsilon r^3 + \text{h.o.t.}.\end{aligned}$$

What happens as ϵ is changed through 0, $\mu_\epsilon = 0$? Assume that

$$\left.\frac{d\mu_\epsilon}{d\epsilon}\right|_{\epsilon=0} \neq 0.$$

There are two cases, where $d\mu_\epsilon/d\epsilon > 0$ and $d\mu_\epsilon/d\epsilon < 0$. In the first case, for $\epsilon < 0$, 0 is a spiral sink, and for $\epsilon > 0$, 0 is a spiral source. Also, the behavior at $\epsilon = 0$ depends on β_ϵ: If $\beta_0 < 0$, then for ϵ near 0, $\beta_\epsilon < 0$, so that means that at $\epsilon = 0$, 0 is a spiral sink.

Now let us look at the global picture at $\epsilon = 0$. There exists an attractor block around 0. Look at

$$\dot{r} = \mu_\epsilon r + \beta_\epsilon r^3.$$

The fixed points of $\mu_\epsilon r + \beta_\epsilon r^3 = 0$ are $r = 0, \pm\sqrt{-\mu_\epsilon/\beta_\epsilon}$. In two dimensions,

Do we get the same picture with higher order terms? The answer is yes. Set up regions in the (r, θ) plane:

Show that near 0, even with higher order terms, r is increasing. Far from 0, r is decreasing. Inside the ring show that the derivative of flow in r direction is < 1. Look at Poincaré return map to $\theta = 0$, show derivative < 1.

Three other pictures:

Global picture doesn't change for small changes in ϵ (r^3 term in $\dot{r}$ is huge when r is big). Behavior near $r = 0$ is changed by changing ϵ. So the rôle of the attractor is transferred from the fixed point to the periodic orbit.

One other case: $\beta_0 = 0 = \beta_\epsilon$ for all ϵ

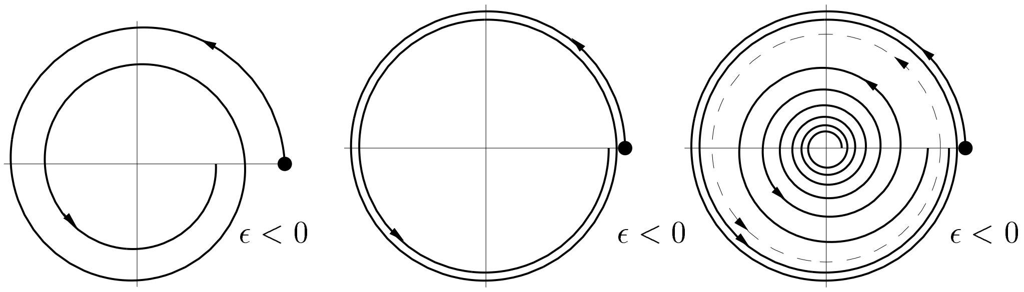

FIGURE 1. For $\epsilon < 0$, we have a spiral sink. When ϵ is 0, we do not have a center, for the nonlinear term makes this still into a spiral sink. And for some value of $\epsilon > 0$, a periodic orbit is born, which is an attractor.

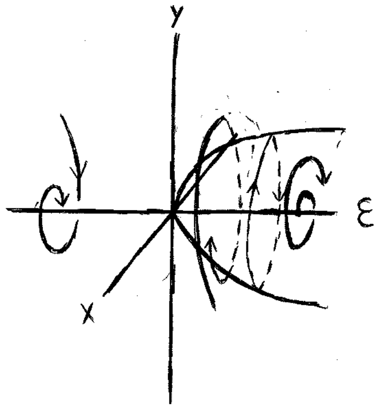

FIGURE 2. The same deal pictured in higher dimensions. The spiral sink at the origin becomes a spiral source, and the attracting cycle is born.

Then

$$\dot{r} = \mu_\epsilon r$$
$$\dot{\theta} = \eta_\epsilon + \gamma_\epsilon r^2, \qquad \left.\frac{d\mu_\epsilon}{d\epsilon}\right|_{\epsilon=0} > 0.$$

Think of β_0 as taking the plane of periodics at $\epsilon = 0$ and bending it.

Now, how can periodic orbits bifurcate? Bifurcation of periodic orbits of flows corresponds to bifurcations of fixed points of maps. Given a periodic orbit, we can associate a Poincaré map.

Look at the 1-parameter family of maps $P_\epsilon \colon \mathbf{R}^n \to \mathbf{R}^n$ with $P_0(0) = 0$.

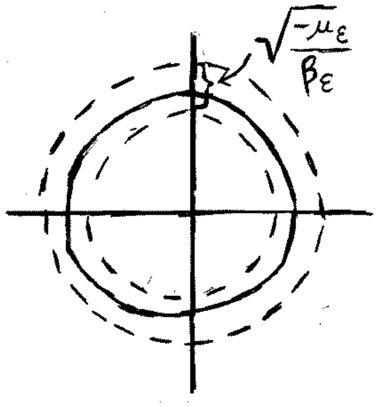

FIGURE 3. the ring

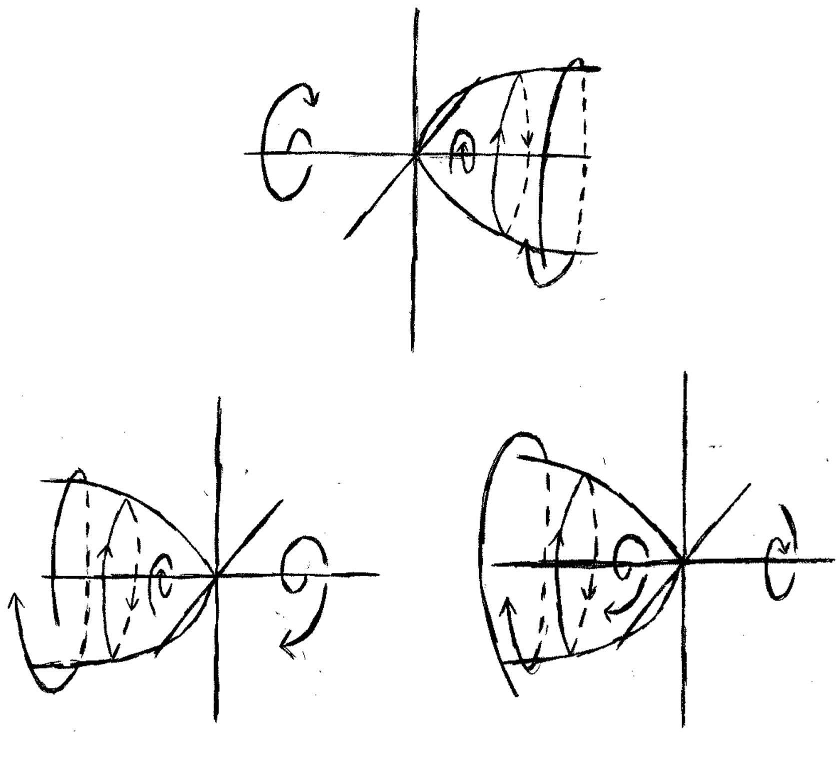

FIGURE 4. The other three possibilities.

THEOREM 1.1. 1. *Suppose the eigenvalues of* $DP_0|_0$ *are off of the unit circle. Then there is a neighborhood* V *of* 0, $\epsilon_1 > 0$, *and* $\gamma\colon (-\epsilon_1, \epsilon_1) \to V$ *as smooth as* P_ϵ *such that*

(a) $\gamma(0) = 0$.

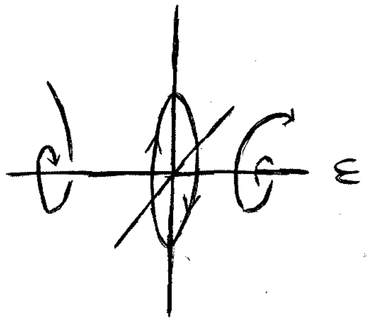

FIGURE 5. The spiral sink becomes a center which becomes a spiral source.

(b) $\forall \epsilon \in (-\epsilon_1, \epsilon_1), P_\epsilon(\gamma(\epsilon)) = \gamma(\epsilon)$
(c) *if* $x \in V$, *and* $\epsilon \in (-\epsilon_1, \epsilon_1)$ *and* $P_\epsilon(x) = x$, *then* $x = \gamma(\epsilon)$.
(d) *The dimensions of* W^u *and* W^s *for* P_ϵ *at* $\gamma(\epsilon)$ *are constant.*

2. *If* $DP_0|_0$ *has no eigenvalues* $= 1$, *then there is* ϵ_1, V, γ *such that the first three conclusions hold.*

Proof: The same as for flows. Define $\mathcal{P}(x, \epsilon) = P_\epsilon(x) - x$ and look at $\mathcal{P}(x, \epsilon) = 0$. To use the Inverse Function Theorem to get γ, we need that $D_x\mathcal{P}|_{(0,0)}$ is non-singular, that is that $D_xP_0 - I$ has no eigenvalues equal to 1. Also note that the eigenvalues of $DP_\epsilon|_{\gamma(\epsilon)}$ are continuous in ϵ. $\square$

So, there are 3 different codimension 1 bifurcations for fixed points of maps:

1. 1 eigenvalue $= 1$ (saddle-node)
2. 1 eigenvalue $= -1$ (period-doubling)
3. Pair of eigenvalues on unit circle (Hopf, Neimark)

2. More Hopf Bifurcation Theorem (Nov. 22,25)

Remember that bifurcations of fixed points of maps correspond to bifurcation of periodic orbits for flows. As we saw last time, there are three distinct types of bifurcations:

1. Saddle node bifurcation (center manifold)

   It suffices to consider the system $P_\epsilon \colon \mathbf{R} \to \mathbf{R}$. **Example:** $P_\epsilon(x) = \epsilon + x$. For $\epsilon = 0$, all points are fixed, but for $\epsilon \neq 0$, there are no fixed points whatsoever.

   We expect that the generic picture should bend this picture one way or the other.

   Look at $P_\epsilon(x) = \epsilon + x + x^2$.

   We need the two nondegeneracy conditions

$$\left.\frac{dP_\epsilon}{d\epsilon}\right|_0 \neq 0, \qquad \left.\frac{d^2P_0}{dx^2}\right|_0 \neq 0.$$

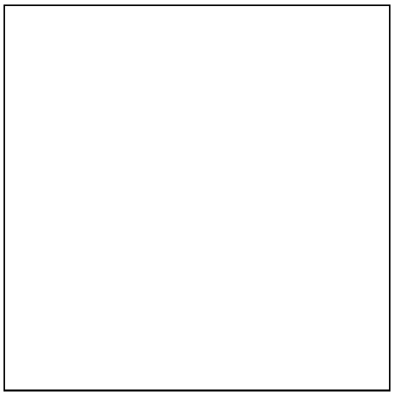

FIGURE 6. For flows, the bifurcations happen as the eigenvalues cross the imaginary axis. For maps, they happen when the eigenvalues cross the unit circle.

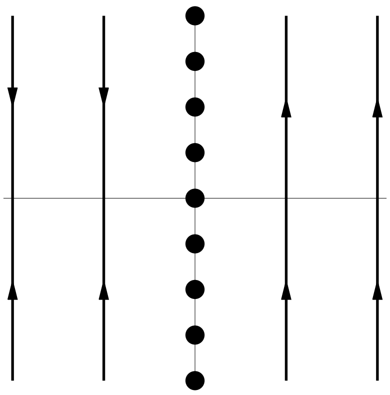

FIGURE 7. A radical bifurcation. No fixed points becomes all fixed points which again becomes no fixed points.

In one dimension, source and sink are born (or die) at bifurcation (at $\epsilon = 0$). In 2 or more dimensions, we expect to have either saddle-sink (if all other eigenvalues of $D_x P_0|_0$ are inside the unit circle), saddle-source (if all other eigenvalues are outside the unit circle), or saddle-saddle.

2. Period-doubling (one eigenvalue $= -1$)

   Simple model, we know $P_0(0) = 0$ and if all other eigenvalues are not on the unit circle, then there is a neighborhood U of 0, $\epsilon_1 > 0$, and a smooth curve $\gamma\colon (-\epsilon_1, \epsilon_1) \to U$ such that if $P_\epsilon(x) = x$, and $x \in U$, then $x = \gamma(\epsilon)$.

   So we may as well assume that $P_\epsilon(0) = 0, \quad \forall \epsilon$. By center manifold it suffices to consider $P_\epsilon\colon \mathbf{R} \to \mathbf{R}$.

   Example: This example is a bit too simple. Consider $P_\epsilon(x) = -(1+\epsilon)x$ For $\epsilon < 0$, 0 is a sink, for $\epsilon > 0$, a source, and for $\epsilon = 0$, every point is periodic of period 2.

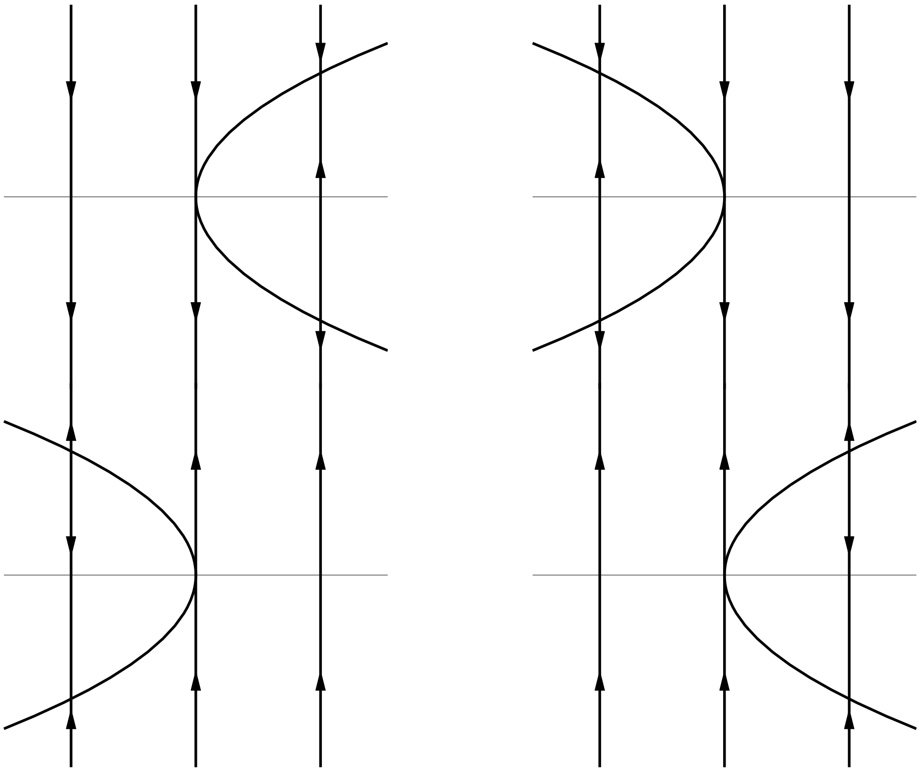

FIGURE 8. These are the possible Hopf bifurcations. These are the same pictures as in Figures 22 and 23.

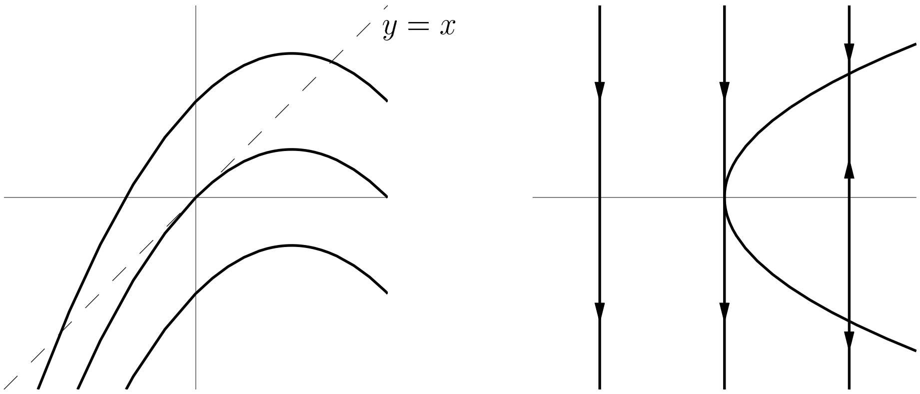

FIGURE 9. This is the map $P_\epsilon(x) = \epsilon + x - x^2$ and its corresponding bifurcation diagram.

Example: $P_\epsilon(x) = -(1+\epsilon)x + x^2$. 0 is a sink for $\epsilon < 0$, and a source for $\epsilon > 0$. We expect the birth of a period 2 orbit at $\epsilon = 0$. Look for fixed points of the 2nd iterate.

$$P_\epsilon^2(x) = -(1+\epsilon)P_\epsilon(x) + (P_\epsilon(x))^2 \quad = (1+\epsilon)^2 x + (\epsilon + \epsilon^2)x^2 - 2(1+\epsilon)x^3 + x^4.$$

Note that at $\epsilon = 0$, P_0^2 has derivative 1 at $x = 0$. Also,

$$P_0^2(x) = x + 0x^2 - 2x^3 + x^4,$$

so it is not a saddle-node bifurcation.

Look for fixed points of $P_\epsilon^2(x)$. $x = 0$ is always fixed. For $\epsilon > 0$, we have the fixed points

$$\frac{\epsilon - \sqrt{4\epsilon + \epsilon^2}}{2}, \text{ and } \frac{\epsilon + \sqrt{4\epsilon + \epsilon^2}}{2}$$

Note that $x = 2 + \epsilon$ far from $x = 0$. The bifurcation for P_ϵ^2 is known as a pitchfork bifurcation.

If λ_ϵ is an eigenvalue with $\lambda_0 = -1$ then we need

$$\frac{d\lambda_\epsilon}{d\epsilon} \neq 0$$

which corresponds to

$$\frac{d^3 P_0^2}{dx^3} \neq 0.$$

Note that $d^2 P_0^2/dx^2$ is always 0 because P_0^2 is not undergoing a saddle-node bifurcation.

The proof for the saddle-node is the same as for flows. Let $\mathcal{P}(x, \epsilon) = P_\epsilon(x) - x$, and try to use the Implicit Function Theorem to write level set of $\mathcal{P} = 0$ as $(x, \gamma(x))$, that is, $\epsilon = \gamma(x)$.

There is one nondegeneracy condition used for the Implicit Function Theorem, i.e.

$$\left.\frac{\partial \mathcal{P}}{\partial \epsilon}\right|_{(0,0)} \neq 0.$$

Other condition forces

$$\left.\frac{d^2\gamma}{dx^2}\right|_{x=0} \neq 0.$$

For period doubling, we try to do the same thing. Let

$$Q(x, \epsilon) = \frac{P_\epsilon^2(x) - x}{x}.$$

Apply Implicit Function Theorem to Q (extend Q smoothly to $x = 0$). The level set of $Q = 0$ is $Q(x, \gamma(x)) = 0$.

Now for the Hopf bifurcation.

Let $F_\epsilon \colon \mathbf{R}^3 \to \mathbf{R}^3$ be a 1-parameter family of vector fields. For all ϵ, $\dot{x} = F_\epsilon(x)$ has a periodic orbit and for $\epsilon = 0$ the eigenvalues of the first return map (the Floquet multipliers) are on the unit circle.

For maps, let $P_\epsilon \colon \mathbf{R}^2 \to \mathbf{R}^2$ be a 1-parameter family, such that $P_\epsilon(0) = 0$, and $DP_0|_0$ has eigenvalues on the unit circle, $e^{i\theta} \neq 1$. How does the stability change as ϵ changes?

Do exactly the same calculation. Complexify and linearize in $\mathbf{C}^2$. On the image of $\mathbf{R}^2$ we only need one of the complex variables (the other is the conjugate).

Write

$$P_\epsilon(z) = \lambda_\epsilon z + \sum_{i+j \leq 2} a_{ij} z^i \bar{z}^j$$

where λ_ϵ is the eigenvalue of DP_ϵ at $x = 0$.

We want to put this in normal form, i.e. get rid of as many higher order terms as possible. To get rid of $z^k\overline{z}^l$ do the change of coordinates

$$w + bw^k\overline{w}^l = z$$

and end up with

$$b = \frac{a_{kl}}{\lambda_\epsilon^k \overline{\lambda}_\epsilon^{\,l} - \lambda_\epsilon}.$$

At $\epsilon = 0$, we get $\lambda_\epsilon = e^{i\theta}$, so the denominators look like

$$e^{ikl}e^{-i\theta l} - e^{i\theta} = e^{i\theta}\left(e^{i(k-l-1)\theta} - 1\right)$$

which must not be 0. So we cannot get rid of terms with $k - l - 1 = 0$. Another case is that if $\theta = 2\pi p/q$, (i.e. $e^{i\theta}$ is a root of unity), we can't have

$$k - l - 1 \equiv 0 \pmod q.$$

The lowest order term where this happens is $k = 0, l = q-1$. So for $\lambda_0 = e^{2\pi ip/q}$, the normal form is

$$P_0(z) = \lambda_0 z + a_1 z^k\overline{z} + c\overline{z}^q.$$

Let us add the assumption that if $\lambda_0 = e^{2\pi ip/q}$ then $q \geq 5$. Then

$$P_\epsilon(z) = \lambda_\epsilon z + a_1 z^2\overline{z} + \text{ h.o.t.}$$

Switching to polar coordinates, if $\lambda_\epsilon = \sigma_\epsilon e^{i\phi_\epsilon}$, then

$$P_\epsilon(re^{i\theta}) = \sigma_\epsilon e^{i\phi_\epsilon re^i\theta} + a_1 r^3 e^{i\theta} + \text{h.o.t.}$$

Let $a_1 = \beta_\epsilon e^{i\gamma_\epsilon}$, then

$$P_\epsilon(re^{i\theta}) = (\sigma_\epsilon r)e^{i(\theta+\phi_\epsilon)} + \beta_\epsilon r^3 e^{i(\theta+\gamma_\epsilon)} + \cdots$$

So

$$P_\epsilon\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} \sigma_\epsilon r + \beta_\epsilon r^3 + \text{h.o.t.} \\ (\theta + \phi_\epsilon) + \mathbf{O}(r^2) + \text{h.o.t.} \end{pmatrix}.$$

Suppose $\|\lambda_\epsilon\| < 1$ for $\epsilon < 0$, and $\|\lambda_\epsilon\| > 1$ for $\epsilon > 0$. The fixed points of $r \mapsto \sigma_\epsilon r + \beta_\epsilon r^3$ are

$$0, \pm\sqrt{\frac{1-\sigma_\epsilon}{\beta_\epsilon}}.$$

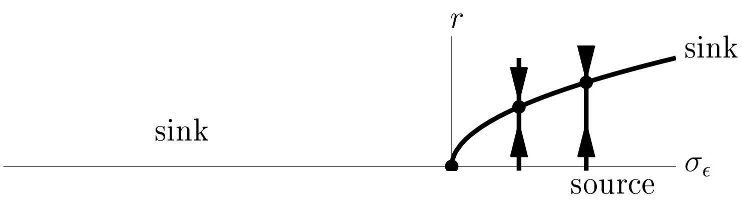

FIGURE 10. The sink becomes a source-sink pair.

Combine r and θ:
Adjusting the signs of $d\sigma_\epsilon/d\epsilon$ for $\epsilon = 0$ and β_ϵ give the other three pictures:

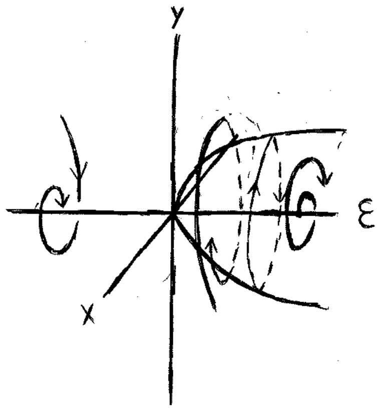

FIGURE 11. The sink becomes an attracting cycle.

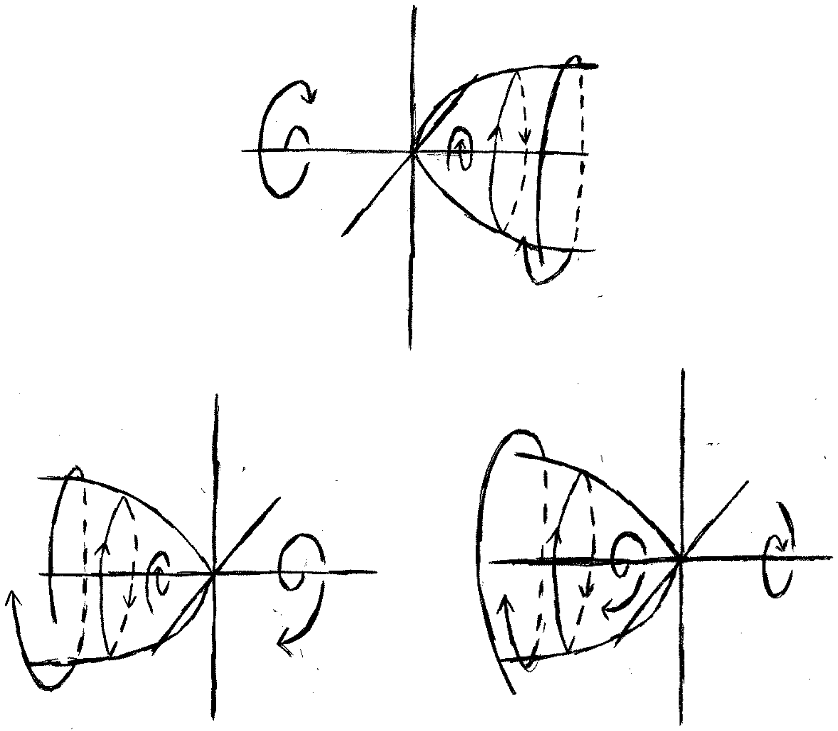

FIGURE 12. Depending on the signs of $d\sigma_\epsilon/d\epsilon$ and β_ϵ, we may have one of these three pictures.

One can show (by building the attractor and repeller blocks) that if the higher order terms are added in, for ϵ near 0, the picture is the same.

The exceptional cases are $\lambda_0 = e^{2\pi i p/q}$ for $q = 1, 2, 3, 4$. If $q = 1$, then $\lambda_0 = 1$, and this is a saddle-node bifurcation, provided only one eigenvalue is ± 1. If $q = 2$, then $\lambda_0 = -1$, and this is a period-doubling bifurcation, again provided only one eigenvalue is ± 1. The cases $q = 3$ and $q = 4$ are special.

3. Setup for Melnikov method (Nov. 25, Dec 2)

Up this point, every theorem that we've had has had the words "for ϵ small". Everything has been local, with one exception: in the Stable-Unstable Manifold Theorem, we get the result that they are globally immersed submanifolds.

The reason this is is that all of our techniques so far have used calculus, so it must be local. We start with something we know, and perturb away. Also, all we have worked with are fixed points, and periodic orbits (which are themselves fixed points of maps). But we can also do local analysis near bigger things, e.g. whole nontrivial orbits.

Let $\dot{x} = F_\epsilon(x, t)$ be a 1-parameter family. Let us assume that we understand an orbit for $\epsilon = 0$ (could be a W^u or a W^s). Then we compute what happens for $\epsilon \neq 0$ (but small).

Definition 3.1. *If there is a point $z \neq x_0$ such that $z \in W^u(x_0) \cap W^s(x_0)$ then z is called a **homoclinic point**. $\mathcal{O}(z)$ is called a **homoclinic orbit**.*

Note: In 2 dimensions, we have either that $W^u(x_0) \cap W^s(x_0) = \{x_0\}$ or that a branch of $W^s(x_0)$ is the same as a branch of $W^u(x_0)$.

Example: $\ddot{y} = y - y^3$. Convert this to the system

$$\begin{aligned}\dot{y} &= v \\ \dot{v} &= y - y^3.\end{aligned}$$

This system is Hamiltonian, with

$$H(v, y) = \frac{v^2}{2} - \frac{y^2}{2} + \frac{y^4}{4}$$

Note that $W^u(0) = W^s(0)$.

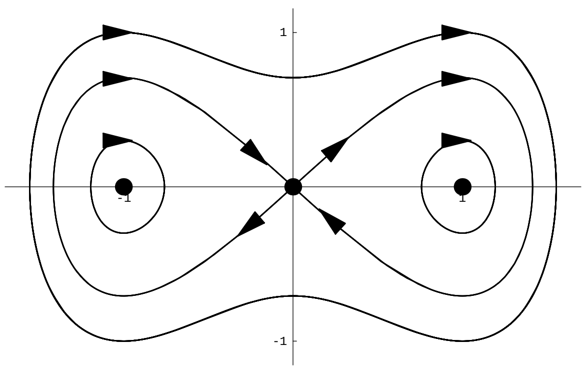

Figure 13. The phase plane for $\ddot{y} = y - y^3$.

If we perturb and get

$$\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ y - y^3 \end{pmatrix} + \epsilon G \begin{pmatrix} y \\ v \end{pmatrix}$$

for ϵ small, we can break the connection (adding damping or antidamping) by letting

$$G \begin{pmatrix} y \\ v \end{pmatrix} = \pm \begin{pmatrix} 0 \\ v \end{pmatrix}.$$

FIGURE 14. This should be a picture of the phase planes of the damped and anti-damped pendula.

Example: Consider the system $\ddot{y} = -\sin y$, or

$$\begin{aligned} \dot{y} &= v \\ \dot{v} &= -\sin y, \end{aligned}$$

this is also Hamiltonian, with

$$H(y, v) = \frac{v^2}{2} - \cos(y)$$

This is a "heteroclinic connection", i.e. a branch of $W^s(\pi, 0)$ is the same as a branch of $W^u(-\pi, 0)$. We can again break the connection with damping or antidamping.

In the 2-dimensional autonomous case, the W^s and W^u manifolds are made up of 3 orbits (fixed point, saddle) and 2 branches. If $z \in W^s(x_0) \cap W^u(x_0)$, then $\mathcal{O}(z) \subseteq W^s(x_0) \cap W^u(x_0)$.

What about the nonautonomous case, e.g.

$$\dot{\begin{pmatrix} y \\ v \end{pmatrix}} = \begin{pmatrix} v \\ y - y^3 \end{pmatrix} + \epsilon G(y, v, t)$$

where $G(y, v, t + T) = G(y, v, t)$. Make this system 3-dimensional:

$$\dot{\begin{pmatrix} y \\ v \\ s \end{pmatrix}} = \begin{pmatrix} v \\ y - y^3 \\ 1 \end{pmatrix} + \begin{pmatrix} \epsilon G(y, v, s) \\ 0 \end{pmatrix}.$$

FIGURE 15. This will be a picture of the phase plane for $v^2/2 - \cos(y)$.

We only need $s \in [0, T]$ because G is periodic. So our new phase space is $\mathbf{R}^2 \times [0, T]$ with 0 and T identified.

For $\epsilon = 0$, there are 3 period-T periodic orbits.

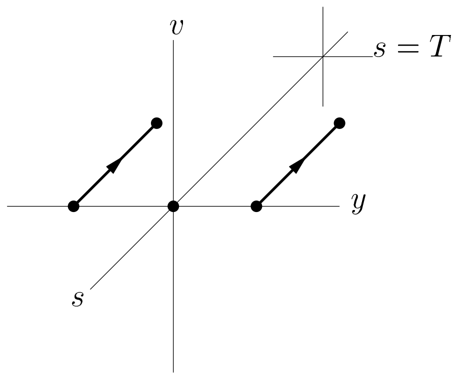

FIGURE 16. The Floquet multipliers of $\mathcal{O}(0,0,0)$ make it a saddle (the other two fixed points are centers).

First, draw the Poincaré map for $s = 0$. This is the period T map of the 2-dimensional flow.

The 3-dimensional picture:

Now make $\epsilon > 0$. Damping or antidamping is still possible, with $G(y, v, t) = (0, \pm v)$.

Remark: By bifurcation theory, we know that if ϵ is small then the Poincaré map has a fixed point near $(0, 0)$.

Another (new) possibility:

FIGURE 17. This will be a picture of the Poincaré map for the unperturbed flow.

FIGURE 18. This will be the phase plane for the flow corresponding to the map in the last figure.

We could have that $z \in W^u(\gamma_\epsilon) \cap W^s(\gamma_\epsilon)$, so $\mathcal{O}(z) \subseteq W^u(\gamma_\epsilon) \cap W^s(\gamma_\epsilon)$. All we know is that the manifolds go through these points, but the picture could be:

This is a picture of a "transverse homoclinic orbit" with z being the transverse homoclinic point, because W^u and W^s cross transversally, i.e. not tangentially. The 3-dimensional picture:

In this setup, a transverse homoclinic point gives us 3 things:

1. infinitely many homoclinic orbits,
2. very nontrivial recurrence,
3. "escape" orbits,

i.e. "chaos". In this case, note that W^u and W^s for the Poincaré map are only immersed (not embedded).

FIGURE 19. A 3-dimensional picture of the flow, with variables y, v, t.

FIGURE 20. The Poincaré map, and the stable and unstable manifolds for the damped pendulum.

These are interesting flows. Are there specific examples? Start with

$$\begin{aligned}\dot{y} &= v\\ \dot{v} &= y - y^3\end{aligned}$$

and devise a perturbation that creates transverse homoclinics,

$$\dot{\begin{pmatrix} y \\ v \end{pmatrix}} = \begin{pmatrix} v \\ y - y^3 \end{pmatrix} + \epsilon G(y, v, t).$$

Look at the Poincaré map, $P \colon \mathbf{R}^2 \to \mathbf{R}^2$, and pick a neighborhood V as shown.

Choose $Q_\epsilon \colon \mathbf{R}^2 \to \mathbf{R}^2$, such that $Q_\epsilon = I$ outside V, and is a small rotation well inside V. Then $Q_\epsilon \circ P$ creates a transverse homoclinic orbit.

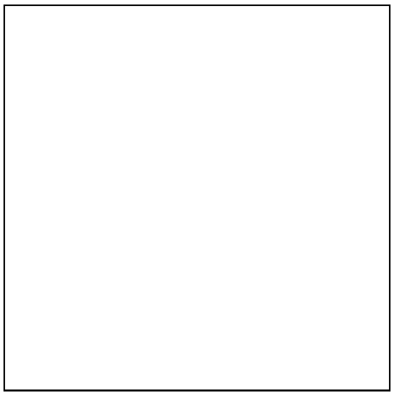

FIGURE 21. The Poincaré map in the case of a homoclinic orbit

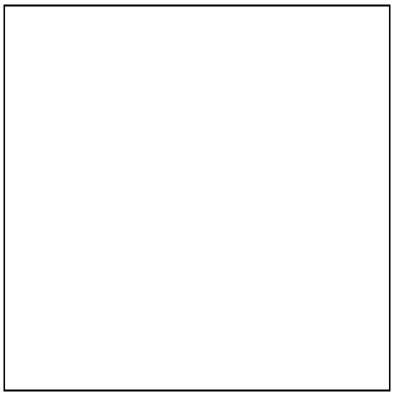

FIGURE 22. A transverse homoclinic crossing

4. Using Melnikov for existence of a transverse homoclinic (Dec. 4)

Our goal is to show a given system has (transverse) homoclinic points. The following is the outline of a method, rather than a theorem. Again, assume

$$\dot{\begin{pmatrix} y \\ v \end{pmatrix}} = F(y, v) + \epsilon G(v, y, t),$$

where $G(y, v, t) = G(y, v, t + T)$. Further assume that when $\epsilon = 0$, F (the unperturbed system) has a saddle, x_0, and has a homoclinic connection, where a branch of $W^u(x_0)$ is a branch of $W^s(x_0)$.

Notation.

1. Let $q_0 : \mathbf{R} \to \mathbf{R}^2$ be an orbit which equals this homoclinic connection. Let $z = q_0(0)$. $q_0(t)$ is a solution for F.

FIGURE 23. A 3-dimensional picture of a transverse homoclinic orbit

FIGURE 24. How to make a transverse homoclinic orbit out of any homoclinic orbit

2. When $\epsilon \neq 0$, think of the system in $\mathbf{R}^2 \times [0, T]$ with 0 and T identified. Call the third variable t.
3. For $\epsilon \neq 0$, but close to 0, the system

$$\dot{\begin{pmatrix} y \\ v \end{pmatrix}} = F(y, v) + \epsilon G(y, v, t)$$

has a period T orbit near $y = v = 0$. Call it γ_ϵ.

What happens when we change ϵ slightly? To see if $W^s(\gamma_\epsilon)$ and $W^u(\gamma_\epsilon)$ cross, try to measure the distance between them.

Steps:

1. Pick $z = q_0(0)$ on unperturbed connecting orbit,
2. build Σ transverse to $W^u(\gamma_\epsilon) = W^s(\gamma_\epsilon)$,
3. Look at $W^u(\gamma_\epsilon) \cap \Sigma$ and $W^s(\gamma_\epsilon) \cap \Sigma$. There are 3 possibilities:

FIGURE 25. The three-dimensional homoclinic

FIGURE 26. The three possible configurations of $W^s(\gamma_\epsilon)$ and $W^u(\gamma_\epsilon)$

4. Try to measure the distance between $W^u(\gamma_\epsilon) \cap \Sigma$ and $W^s(\gamma_\epsilon) \cap \Sigma$ as a function of t. Let $d(t, \epsilon)$ be the distance between $W^s(\gamma_\epsilon) \cap \Sigma \cap \{t = t_0\}$ and $W^u(\gamma_\epsilon) \cap \Sigma \cap \{t = t_0\}$.
5. Write $d(t, \epsilon)$ as $d_0(t) + \epsilon d_1(t) + \epsilon^2 d_2(t) + \cdots$ (assume that we can). When $\epsilon = 0$, $W^u = W^s$, so $d_0(t) = 0$, so

$$d(t, \epsilon) = \epsilon d_1(t) + \epsilon^2 d_2(t) + \cdots$$

What we'll actually compute is $d_1(t)$.

LEMMA 4.1. *If $d(t, \epsilon) = \epsilon d_1(t) + \mathbf{O}(\epsilon^2)$ stays bounded, and $d(t+T, \epsilon) = d(t, \epsilon)$ for all t, then*

(a) *If $d_1(t) > 0$ for all t, then for ϵ sufficiently small, $d(t, \epsilon) > 0$ for all t.*
(b) *If $d_1(t) < 0$ for all t, then for ϵ small, $d(t, \epsilon) < 0$ for all t.*
(c) *If $d_1(t_1) > 0, d_1(t_2) < 0$, then $\exists t_\epsilon$ such that for ϵ sufficiently small, $d(t_\epsilon, \epsilon) \neq 0$.*

FIGURE 27. The geometric interpretation of $d(t, \epsilon)$

(d) *If* $d_1(t_0) = 0, d'(t_0) \neq 0$, *then* $\exists t_\epsilon$ *for* ϵ *sufficiently small such that* $d(t_\epsilon, \epsilon) = 0$ *and*

$$\frac{\partial d(t_\epsilon, \epsilon)}{\partial t} \neq 0.$$

6. Compute $d_1(t)$, which is the order ϵ term of the signed distance between $W^s(\gamma_\epsilon)$ and $W^u(\gamma_\epsilon)$ on Σ. Then check if $d_1(t)$ is ever 0. If it is, then for ϵ sufficiently small, the system has homoclinic points.

Now why do we hope that we can compute $d_1(t)$? Look at system

$$\dot{\begin{pmatrix} y \\ v \end{pmatrix}} = F(y, v) + \epsilon G(y, v, t).$$

We know that $q_0(t)$ is a solution for $\epsilon = 0$, and $q_0(t) \to (0, 0)$ as $t \to \pm\infty$. When $\epsilon \neq 0$, but small, try to write solutions near $q_0(t)$, using Euler's method:

FIGURE 28. The errors in Euler's Method

If we follow the vector field for $\epsilon \neq 0$, the first step error is of order $\epsilon \cdot \Delta t$, where Δt is the step size. There are two sources of error on the next step:

1. Because the red orbit is off of $q_0(t)$, the orbits can move exponentially apart. This error is of order ϵ and fast-growing.

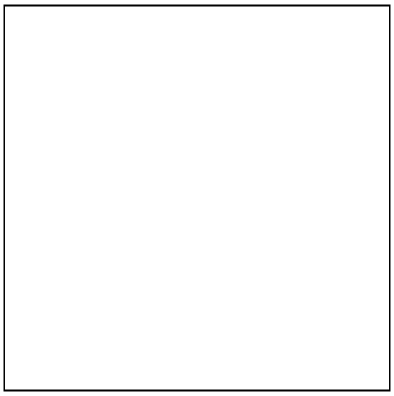

FIGURE 29. How fast can order-ϵ error grow?

Remark: In the example systems, we don't have exponential divergence.

2. Same type of error as first step (we changed vector field by ϵ so there is a $\Delta t \cdot \epsilon$ change in the next step).
3. Compounding of change in vector field, but this error is order ϵ^2.

We expect $d_1(t)$ to have 2 terms:

1. For divergence of nearby orbits if $\epsilon = 0$,
2. ϵ-order changes along $q_0(t)$.

We need two tools:

1. A natural way to pick Σ. We will use $F^{\perp}(z)$ to choose Σ. At each point of q_0, measure the perturbation in the $F^{\perp}$ direction.
2. A nice representation of $W^s(\gamma_\epsilon)$ and $W^u(\gamma_\epsilon)$.

5. Calculation of order ϵ term (Dec. 6)

So, we chose Σ to be a surface in $\mathbf{R}^3$ through $q_0(0) \times [0, T]$ containing $F^{\perp}(z)$. We need a way to name orbits on $W^s(\gamma_\epsilon)$,$W^u(\gamma_\epsilon)$.

Let $q^s_\epsilon(\cdot, t_0) \colon \mathbf{R}^2 \to \mathbf{R}^2$ be a solution of

$$\dot{\begin{pmatrix} y \\ v \end{pmatrix}} = F(y, v) + \epsilon G(y, v, t),$$

and $q^s_\epsilon(t, t_0) \to \gamma_\epsilon$ as $t \to +\infty$ (so q^s_ϵ is on $W^s(\gamma_\epsilon)$) and $(q^s_\epsilon(t_0, t_0), t_0) \in \Sigma$ and for all $t \geq t_0$, $(q^s_\epsilon(s, t_0), s) \notin \Sigma$. Do the same for $q^u_\epsilon(t, t_0)$, with $(q^u_\epsilon(t_0, t_0), t_0) \in \Sigma$ and $(q^u_\epsilon(s, t_0), s) \notin \Sigma$ for all $s < t_0$.

We are interested in measuring $\|q^s_\epsilon(t_0, t_0) - q^u_\epsilon(t_0, t_0)\| = d(t_o, \epsilon)$.

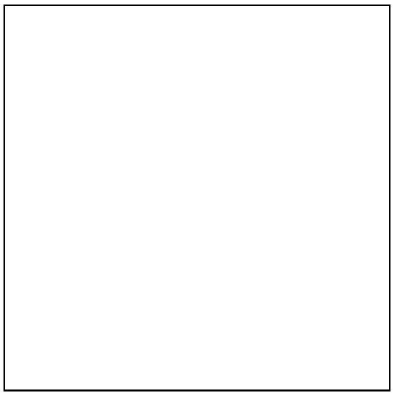

FIGURE 30. $F(z)$ and $F^{\perp}(z)$

LEMMA 5.1. *As* $\epsilon \to 0$, $q_\epsilon^s, q_\epsilon^u \to q_0$. *We can write*

$$q_\epsilon^s(t,t_0) = q_0(t-t_0) + \epsilon q_1^s(t,t_0) + \mathbf{O}(\epsilon^2)$$
$$q_\epsilon^u(t,t_0) = q_0(t-t_0) + \epsilon q_1^u(t,t_0) + \mathbf{O}(\epsilon^2).$$

This is a corollary of the Stable/Unstable Manifold theorem.
Note: $q_0(t-t_0) = q_0(0)$ when $t = t_0$.
We want to compute

$$\begin{aligned}
F^{\perp}(z) \cdot [q_\epsilon^s(t_0,t_0) - q_\epsilon^u(t_0,t_0)]& \\
&= F^{\perp}(z) \cdot \big([q_0(t_0-t_0) + \epsilon q_1^s(t_0,t_0)] - [q_0(t_0-t_0) + \epsilon q_1^u(t_0,t_0)] + \mathbf{O}(\epsilon^2)\big) \\
&= \epsilon \underbrace{F^{\perp}(z) \cdot [q_1^s(t_0,t_0) - q_1^u(t_0,t_0)]}_{\text{compute this!}} + \mathbf{O}(\epsilon^2)
\end{aligned}$$

Now,

$$\begin{aligned}
F^{\perp}(z) \cdot [q_1^s(t_0,t_0) - q_1^u(t_0,t_0)]& \\
&= F^{\perp}(q_0(t_0-t_0)) \cdot [q_1^s(t_0,t_0) - q_1^u(t_0,t_0)].
\end{aligned}$$

This is the value at $t = t_0$ of the function

$$F^{\perp}(q_0(t-t_0)) \cdot [q_1^s(t,t_0) - q_1^u(t,t_0)].$$

Compute d/dt of this and hope it turns out nice. First note that

$$q_\epsilon^s(t,t_0) = q_0(t-t_0) + \epsilon q_1^s(t,t_0) + \mathbf{O}(\epsilon^2),$$

so

$$\dot{q}_\epsilon^s(t,t_0) = \dot{q}_0(t-t_0) + \epsilon \dot{q}_1^s(t,t_0) + \mathbf{O}(\epsilon^2).$$

We know that q_ϵ^s is a solution of the differential equation, so

$$\begin{aligned}
\frac{d}{dt}(q_\epsilon^s(t,t_0)) &= F(q_\epsilon^s(t,t_0)) + \epsilon G(q_\epsilon(t,t_0),t) \\
&= F(q_0(t-t_0) + \epsilon q_1^s(t,t_0) + \mathbf{O}(\epsilon^2)) \\
&\quad + \epsilon G(q_0(t-t_0) + \epsilon q_1^s(t,t_0) + \mathbf{O}(\epsilon^2),t) \\
&= F(q_0(t-t_0)) + \epsilon \, DF|_{q_0(t-t_0)} \, q_1^s(t,t_0) + \epsilon G(q_0(t-t_0),t).
\end{aligned}$$

So we get

$$\begin{aligned}
&\dot{q}_0(t-t_0) + \epsilon \dot{q}_1^s(t,t_0) + \mathbf{O}(\epsilon^2) \\
&\qquad = F(q_0(t-t_0)) + \epsilon \, DF|_{q_0(t-t_0)} \, q_1^s(t,t_0) + \epsilon G(q_0(t-t_0),t) + \mathbf{O}(\epsilon^2),
\end{aligned}$$

so

$$\epsilon \dot{q}_1^s(t,t_0) = \epsilon \, DF|_{q_0(t-t_0)} \, q_1^s(t,t_0) + \epsilon G(q_0(t-t_0),t) + \mathbf{O}(\epsilon^2),$$

since $F(q_0(t-t_0)) = \dot{q}_0(t-t_0)$.

Thus

$$\dot{q}_1^s(t,t_0) = \underbrace{DF|_{q_0(t-t_0)} \, q_1^s(t,t_0)}_{\text{term 1}} + \underbrace{G(q_0(t-t_0),t)}_{\text{term 2}}.$$

Term 1 involves the rate of change of F, where the small error grows exponentially, and term 2 is an order ϵ perturbation along the unperturbed orbit.

Now look at

$$\Delta^s(t,t_0) = F^\perp(q_0(t-t_0)) \cdot q_1^s(t,t_0).$$

$$\begin{aligned}
\dot{\Delta}^s(t,t_0) &= DF|_{q_0(t-t_0)} \, \dot{q}_0(t-t_0) \cdot q_1^s(t,t_0) + F^\perp(q_0(t-t_0))\dot{q}_1^s(t,t_0) \\
&= DF^\perp\big|_{q_0(t-t_0)} \, F(q_0(t-t_0)) \cdot q_1^s(t,t_0) \\
&\quad + F^\perp(q_0(t-t_0)) \cdot \Big(DF|_{q_0(t-t_0)} \, q_1^s(t,t_0) + G(q_0(t-t_0),t \Big),
\end{aligned}$$

or

$$\begin{aligned}
\dot{\Delta}^s(t,t_0) = &\Big[DF^\perp\big|_{q_0(t-t_0)} \cdot F(q_0(t-t_0)) + F^\perp(q_0(t-t_0)) \cdot DF|_{q_0(t-t_0)} \Big] \cdot q_1^s(t,t_0) \\
&+ F^\perp(q_0(t-t_0)) \cdot G(q_0(t-t_0),t_0).
\end{aligned}$$

So if we know $q_0(t-t_0)$, then we get a hard ODE. In the example

$$\begin{aligned}
\dot{y} &= v \\
\dot{v} &= y - y^3,
\end{aligned}$$

we see that we don't always have exponential splitting.

Note:

$$\begin{aligned}
&DF^\perp\big|_{q_0(t-t_0)} \cdot F(q_0(t-t_0)) + F^\perp(q_0(t-t_0)) \cdot DF|_{q_0(t-t_0)} \\
&\quad = \operatorname{trace}\Big(DF|_{q_0(t-t_0)} \cdot F^\perp(q_0(t-t_0)) \Big)
\end{aligned}$$

Let

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \qquad F^\perp = \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}.$$

so

$$DF = \begin{pmatrix} \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial v} \end{pmatrix}, \qquad DF^\perp = \begin{pmatrix} -\dfrac{\partial f_2}{\partial y} & -\dfrac{\partial f_2}{\partial v} \\ \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial v} \end{pmatrix}.$$

trace DF measures the rate of growth of pieces of area under the flow F. Our example was Hamiltonian, with

$$F = \left(\frac{\partial H}{\partial v}, -\frac{\partial H}{\partial y}\right),$$

so

$$DF = \begin{pmatrix} \dfrac{\partial^2 H}{\partial y \partial v} & \dfrac{\partial^2 H}{\partial v^2} \\ -\dfrac{\partial^2 H}{\partial y^2} & -\dfrac{\partial^2 H}{\partial v \partial y} \end{pmatrix},$$

so $\operatorname{trace}(DF) = 0$!

So let us assume that F is Hamiltonian (or at least that $\operatorname{trace}(DF) = 0$), so

$$\dot{\Delta}^s(t, t_0) = F^\perp(q_0(t - t_0)) \cdot G(q_0(t - t_0), t),$$

giving

$$\Delta^s(t, t_0) = \int F^\perp(q_0(t - t_0)) \cdot G(q_0(t - t_0), t)\, dt.$$

Note: $F^\perp(q_0(t - t_0)) \to 0$ as $t \to \pm\infty$.

So

$$\frac{d}{dt}\Delta^s(t_0, t_0) = -\int_{t_0}^{\infty} F^\perp(q_0(t - t_0)) \cdot G(q_0(t - t_0), t)\, dt.$$

The above, and

$$\frac{d}{dt}\Delta^u(t, t_0) = F^\perp(q_0(t - t_0)) \cdot q_1^u(t - t_0)$$

gives

$$\Delta^s(t, t_0) = \int_{-\infty}^{t_0} F^\perp(q_0(t - t_0)) \cdot G(q_0(t - t_0), t)\, dt$$

by the same calculation.

So $q_0(0) = q_0(t_0 - t_0)$. Putting the two calculations together, we have

$$\begin{aligned} F^\perp(z) \cdot [q_1^s(t_0, t_0) - q_1^u(t_0, t_0)] & \\ = \Delta^s(t_0, t_0) - \Delta^u(t_0, t_0) & \\ = -\int_{t_0}^{\infty} F^\perp(q_0(t - t_0)) \cdot G(q_0(t - t_0), t)\, dt & \\ - \int_{-\infty}^{t_0} F^\perp(q_0(t - t_0)) \cdot G(q_0(t - t_0), t)\, dt & \\ = -\int_{-\infty}^{\infty} F^\perp(q_0(t - t_0)) \cdot G(q_0(t - t_0), t)\, dt. & \end{aligned}$$

So we have calculated the order ϵ term of the distance between $W^s(\gamma_\epsilon)$ and $W^u(\gamma_\epsilon)$ on Σ.

6. An example of a Melnikov calculation (Dec. 9)

THEOREM 6.1. *Assume a Hamiltonian system*

$$\dot{\begin{pmatrix} y \\ v \end{pmatrix}} = \begin{pmatrix} \partial H/\partial v \\ -\partial H/\partial y \end{pmatrix} = F(y,v),$$

with saddle x_0 and homoclinic orbit $q_0(t)$.

For the perturbed system

$$\dot{\begin{pmatrix} y \\ v \end{pmatrix}} = F(y,v) + \epsilon G(y,v,t)$$

where for all t, $G(y,v,t+T) = G(y,v,t)$, let γ_ϵ denote the hyperbolic periodic orbit near x_0.

Let

$$\begin{aligned} M(t_0) &= \int_{-\infty}^{\infty} F^{\perp}(q_0(t-t_0)) \cdot G(q_0(t-t_0),t)\,dt \\ &= \int_{-\infty}^{\infty} F^{\perp}(q_0(\tau)) \cdot G(q_0(\tau),\tau+t_0)\,d\tau. \end{aligned}$$

This is known as the ***Melnikov integral****.*
Remark: *$M(t)$ is periodic with period T.*
Then

1. *If $M(t_0) \neq 0$ for all t_0, then for ϵ small, γ_ϵ has no homoclinic orbits.*
2. *If $M(t_0) > 0$ for some t_0,$M(t_1) > 0$ for some t_1, then for small ϵ, $W^s(\gamma_\epsilon) \cap W^u(\gamma_\epsilon)$ contains more than γ_ϵ.*
3. *If $M(t_0) = 0$, $M'(t_0) \neq 0$ for some t_0 then for small ϵ, $W^s(\gamma_\epsilon)$ intersects $W^s(\gamma_\epsilon)$ transversally, i.e. there is a transverse homoclinic orbit.*

Example: [Guckenheimer and Holmes, p.191] Consider

$$\begin{aligned} \dot{y} &= v \\ \dot{v} &= y - y^3 + \epsilon(\gamma\cos(\omega t) - \delta v) \end{aligned}$$

For $\epsilon = 0$, the Hamiltonian is

$$H(y,v) = \frac{v^2}{2} - \frac{y^2}{2} + \frac{y^4}{4}.$$

There is a saddle at $(0,0)$ with a homoclinic connection. Solving H we get

$$v = \pm y\sqrt{1 - \frac{y^2}{2}}$$

and we see that

$$q_0(t) = (\sqrt{2}\,\mathrm{sech}\,t, -\sqrt{2}\,\mathrm{sech}\,t\tanh t).$$

The perturbation has 4 parameters: δ is the damping term, γ is the periodic forcing, ω is the frequency, and ϵ is the total size of the perturbation. So let us fix δ, γ, ω.

The Melnikov integral is

$$\begin{aligned}
M(t_0) &= \int_{-\infty}^{\infty} (F^{\perp}(q_0(t)) \cdot G(q_0(t), t+t_0))\, dt \\
&= \int_{-\infty}^{\infty} \left(\sqrt{2}\operatorname{sech} t - (\sqrt{2}\operatorname{sech} t)^3, (\sqrt{2}\operatorname{sech} t, \tanh t)\right) \cdot \\
&\qquad \cdot \left(0, \gamma\cos(\omega(t+t_0)) - \delta(-\sqrt{2}\operatorname{sech} t \tanh t)\right)\, dt \\
&= \int_{-\infty}^{\infty} \sqrt{2}\gamma \operatorname{sech} t \tanh t \cos(\omega(t-t_0))\, dt \\
&\qquad + \int_{-\infty}^{\infty} 2\delta \operatorname{sech}^2 t \tanh^2 t\, dt \\
&= -\frac{4\delta}{3} - \sqrt{2}\gamma\pi\omega \operatorname{sech}\left(\frac{\pi\omega}{2}\right) \sin(\omega t_0) \\
&= -k_1 - k_2 \sin(\omega t_0)
\end{aligned}$$

where $k_1 = 4\delta/3, k_2 = \sqrt{2}\gamma\pi \operatorname{sech}(\pi\omega/2)$.

If $\delta = 0$, then $M(t_0)$ does change sign transversally, so there are homoclinics. If $|k_2| < |k_1|$ then no homoclinics for small ϵ, but if $|k_2| > |k_1|$, then there are homoclinics for sufficiently small ϵ.

Note: As $\omega \to \infty$,

$$\operatorname{sech}\left(\frac{\pi\omega}{2}\right) = \frac{2}{e^{\pi\omega/2} + e^{-\pi\omega/2}},$$

so even small damping will kill the homoclinics, where ω big implies short periodic forcing.

7. Averaging (Dec. 11)

For this section, look at

$$\dot{x} = \epsilon f(x, t, \epsilon)$$

where $x \in \mathbf{R}^n$, and for all t, $f(x, t+T, \epsilon) = f(x, t, \epsilon)$, for ϵ small.

What can we say about this? Almost nothing – because when $\epsilon = 0$, $\dot{x} = 0$. How can $\dot{x} = 0$ bifurcate? Every way!

We might ask: How much does it matter that f depends on t? Let

$$\overline{f}(x) = \frac{1}{T}\int_0^T f(x, t, 0)\, dt$$

be an averaged vector field. Let

$$\tilde{f}(x, t, \epsilon) = f(x, t, \epsilon) - \overline{f}(x).$$

THEOREM 7.1. *If f is C^r, then there is a C^r change of coordinates*

$$x = y + \epsilon w(y, t, \epsilon)$$

such that the system $\dot{x} = \epsilon f(x, t, \epsilon)$ becomes

$$\dot{y} = \epsilon \overline{f}(y) + \epsilon^2 f_1(y, t, \epsilon)$$

where for all t, $f_1(y, t+T, \epsilon) = f_1(y, t, \epsilon)$.

Remark: *This is progress because the nonautonomous part is higher order.*

Note: *This is* **not** *the same as*

$$\dot{y} = \epsilon \overline{f}(y) + \delta f_1(y, t, \epsilon)$$

where δ is arbitrarily small, since above δ is not arbitrarily small, $\delta = \epsilon^2$.

Moreover,

1. *Let $\dot{z} = \epsilon \overline{f}(z)$. If $z(t), y(t)$ are solutions such that $|z(0) - y(0)| < \mathbf{O}(\epsilon)$, then $|z(t) - y(t)| < \mathbf{O}(\epsilon)$ on a time scale $t \sim 1/\epsilon$.*
2. *If z_0 is a hyperbolic fixed point of*
$$\dot{z} = \epsilon \overline{f}(z),$$
then there is an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, there is a periodic orbit $y_\epsilon(t)$ for
$$\dot{y} = \epsilon \overline{f}(y) + \epsilon^2 f_1(y, t, \epsilon)$$
with $y_\epsilon(t) = z_0 + \mathbf{O}(\epsilon)$.
3. *If $z^s(t)$ is in $W^s(z_0)$, $y^s(t)$ is in $W^s(y_\epsilon)$, and $|z^s(0) - y^s(0)| = \mathbf{O}(\epsilon)$, then $|z^s(t) - y^s(t)| = \mathbf{O}(\epsilon)$ for all $t \geq 0$.*

So we can push time dependence to higher order in ϵ. We need to find w in the change of coordinates. Since

$$\begin{aligned} x &= y + \epsilon w(y, t, \epsilon), \\ \dot{x} &= \dot{y} + \epsilon D_y w(y, t, \epsilon) \cdot \dot{y} + \epsilon \frac{\partial w}{\partial t}. \end{aligned}$$

This gives us

$$\begin{aligned} [I + \epsilon D_y w]\, \dot{y} &= \dot{x} - \epsilon \frac{\partial w}{\partial t}, \\ [I + \epsilon D_y w]\, \dot{y} &= \epsilon \overline{f}(x) + \epsilon \tilde{f}(x, t, \epsilon) - \epsilon \frac{\partial w}{\partial t}, \end{aligned}$$

where $\tilde{f}(x, t, \epsilon) = f(x, t, \epsilon) - \overline{f}(x)$.

So

$$\begin{aligned} \dot{y} &= [I + \epsilon D_y w]^{-1} \left[\epsilon \overline{f}(y + \epsilon w) + \epsilon \tilde{f}(y + \epsilon w, t, \epsilon) - \epsilon \frac{\partial w}{\partial t} \right] \\ \dot{y} &= [I + \epsilon D_y w]^{-1} \left[\epsilon \overline{f}(y) + \epsilon \tilde{f}(y, t, \epsilon) - \epsilon \frac{\partial w}{\partial t} + \mathbf{O}(\epsilon^2) \right], \end{aligned}$$

assuming everything is at least $\mathcal{C}^2$.

So take

$$\frac{\partial w}{\partial t} = \tilde{f}(y, t, \epsilon),$$

so

$$w(y, t, \epsilon) = \int \tilde{f}(y, t, \epsilon)\, dt.$$

Note:

$$[I + \epsilon D_y w]^{-1} = I + \mathbf{O}(\epsilon).$$

Then

$$\begin{aligned}\dot{y} &= \epsilon\overline{f}(y) + \mathbf{O}(\epsilon^2)\\ &= \epsilon\overline{f}(y) + \epsilon^2 f_1(y,t,\epsilon).\end{aligned}$$

Compare

$$\begin{aligned}\dot{y} &= \epsilon f(y) + \epsilon^2 f_1(y,t,\epsilon) \text{ to}\\ \dot{z} &= \epsilon\tilde{f}(z).\end{aligned}$$

Let us prove the first part of the theorem:

Proof:

$$\begin{aligned}|z(t)-y(t)| = \Bigg| z(0) - y(0) + \epsilon\int_0^t \overbrace{\left[\overline{f}(z(s)) - \overline{f}(y(s))\right]}^{\leq L|z(s)-y(s)|}\, ds\\ - \epsilon^2\int_0^t f_1(y(s),s,\epsilon)\, ds\Bigg|\end{aligned}$$

So

$$\begin{aligned}|y(t)-z(t)| \leq{}& |z(0)-y(0)|\\ &+ \epsilon\int_0^t \left|\overline{f}(z(s)) - \overline{f}(y(s))\right|\, ds\\ &+ \epsilon^2\int_0^t |f_1(y(s),s,\epsilon)|\, ds\end{aligned}$$

Now let $\zeta(t) = z(t) - y(t)$. Then

$$|\zeta(t)| \leq |\zeta(0)| + \epsilon L\int_0^t |\zeta(s)|\, ds + \epsilon^2\int_0^t C\, dt$$

where L is the Lipschitz constant for $\overline{f}_1$ and C is a bound for f_1 (on neighborhoods $z(s), y(s), 0 \leq s \leq t$.

Remember Grönwall's inequality: if

$$\begin{aligned}&v(t) \leq c(t) + \int_0^t u(s)v(s)\, ds, \text{ then}\\ &v(t) \leq c(0)\exp\left\{\int_0^t u(s)\, ds\right\} + \int_0^t c'(s)\exp\left\{\int_s^t u(\tau)\, d\tau\right\}\, ds.\end{aligned}$$

Let $c(t) = \zeta(0) + \epsilon^2 Ct$, then

$$|\zeta(t)| \leq |\zeta(0)|\, e^{\epsilon Lt} + \epsilon^2 C\int_0^t e^{\epsilon L(t-s)}\, ds.$$

So

$$\begin{aligned}|\zeta(t)| &\leq |\zeta(0)|\, e^{\epsilon Lt} + \frac{\epsilon C}{L}e^{\epsilon tL} - \frac{\epsilon C}{L}\\ &\leq |\zeta(0)|\, e^{\epsilon Lt} + \frac{\epsilon C}{L}e^{\epsilon tL},\end{aligned}$$

so

$$|\zeta(t)| \leq \left||\zeta(0)| + \frac{\epsilon C}{L}\right| e^{\epsilon Lt}.$$

Suppose that $|\zeta(0)| < k\epsilon$ for some k. Then

$$|\zeta(t)| \leq \left|\epsilon k + \frac{\epsilon C}{L}\right| e^{\epsilon L t}.$$

so for $t < 1/\epsilon$,

$$|\zeta(t)| \leq \epsilon \left|\left[k + \frac{C}{L}\right]\right| e^{L}.$$

$\square$

Why look at a system like this? One case may be where you have a center. If you take the right T, the time T map looks like the identity.

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